Journal of Statistical Physics, Vol. 101, Nos. 1/2, 2000

On Two-Dimensional Area-Preserving Diffeomorphisms with Infinitely Many Elliptic Islands

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Received June 30, 1999; final March 15, 2000

We consider two-parameter families of C^r-smooth, $r \geqslant 6$, two-dimensional area-preserving diffeomorphisms that have structurally unstable simplest heteroclinic cycles. We find the conditions when diffeomorphisms under consideration possess infinitely many periodic generic elliptic points and elliptic islands.

KEY WORDS: Area-preserving diffeomorphisms; structurally unstable heteroclinic cycle; elliptic periodic point; elliptic island; the conservative Henon map.

INTRODUCTION

It is well known that role which Poincaré's "New methods of celestial mechanics" have played for formation of the modern theory of nonintegrable Hamiltonian systems. Among the numerous problems posed by Poincaré in this memoir[(1)] one of the first is the following: to prove that models of the classical mechanics possess infinitely many periodic motions. Moreover, he formulates here the stronger hypothesis: periodic motions are dense in the phase spaces of such type models. In this connection, it is very important that Poincaré says about stable periodic motions, since "the especial value of these periodic solutions is explained by the fact that they are the only possible breach across which we could penetrate into the topic considered before as inaccessible." From this point of view, it is quite clear

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0022-4715/00/1000-0321$18.00/0

his position that, for the celestial mechanics, a problem on geodesic flows on convex surfaces,[(2)] where the stable geodesics exist, is more important than a problem on geodesic flows on surfaces with the negative curvature, where all geodesics are unstable. Saying about a stability of periodic motions Poincaré meant the "perpetual stability," i.e., the Lyapunov stability. From contemporary point of view based on the Kolmogorov–Arnol'd–Moser theory (the KAM-theory), one can speak about the Lyapunov stability of periodic motions in Hamiltonian systems only in the case of two degrees of freedom, actually. Here, at general conditions, quaranteering the twisting of a Poincaré map (see refs. 3–5), a fixed point of this map is the elliptic point of the stable type. Such an elliptic point is called *generic*. By the KAM-theory, the fixed generic elliptic point will be enclosed by infinitely many closed invariant curves. Besides, it follows from ref. 6 for the analytical case that, generally speaking, in any neighbourhood of the generic elliptic point there exist infinitely many zones of instability, in the sense of Birkhoff, containing periodic elliptic and hyperbolic points and, moreover, the latters have transverse homoclinic orbits.

In the case of more degrees of freedom, the situation is more complicated. Unlike the two-dimensional case, a generic n-elliptic point (with $n>1$ pair complex conjugate eigenvalues on the unit circle) may be unstable in the usual, (Lyapunov) sense: at some initial conditions near this point the orbits can escape a small neighbourhood of the point due to a mechanism of instability which is called the Arnol'd diffusion. However, for the majority of initial conditions (the initial conditions on invariant tori) the orbit never escapes the small neighborhood of the fixed point.

We note at once, that the Poincaré problem has not been solved until now in any reasonable sense. The attempt of Poincaré, based on the small parameter method, to prove the existence of infinitely many stable (isolated) periodic motions in Hamiltonian systems close to nondegenerate integrable ones was unsuccessful. (This method allows to prove only the existence of a finite number of such orbits, but this number, however, tends to infinity as the small parameter tends to zero). An another approach, based on using the geometrical Poincaré–Birkhoff theorem and the KAM-theory, allows, in principle, to establish the existence of an infinite set of periodic motions, but does not exclude a possibility of their nonisolateness. Naturally, the property of nonisolateness of periodic points is not general. Combinating results of refs. 7 and 8 based on the typicalness[2] of maps in the smooth topology, one can conclude that, in general case, infinitely

[2] A property is called typical (and the systems possessing this property are called typical) if it is carried out for a residual (second category) set of systems, i.e., for such a set which can be presented as a countable intersection of open everywhere dense sets.

many generic elliptic points exist in any neighbourhood of the generic elliptic fixed point.[3]

A weaker variant of the Poincaré problem—to prove the density of periodic motions—has not been solved till now. But the much more progress is achieved here. First of all, it is necessary to mention a number of results about hyperbolic periodic orbits. For example, according to the Birkhoff–Smale–Shilnikov theory, the set of orbits entirely lying in a small neighbourhood of a transverse homoclinic orbit is a hyperbolic set containing infinitely many saddle periodic motions. The same type sets of orbits can be found in levels of a Hamiltonian close to the level containing saddle or saddle-focus equilibria with transverse homoclinic loops.[(9–13)]

In this connection, we note, that for C^1-smooth symplectic diffeomorphisms, given on compact manifolds, the following properties are typical:

(1) Hyperbolic periodic points are dense in the phase space.[(14)]

(2) Every hyperbolic periodic point has a transverse homoclinic point in any neighbourhood of any point of the phase space.[(8, 15, 16)]

(3) If a symplectic diffeomorphism f is not Anosov, then the 1-elliptic points[4] of f are dense in the phase space.[(16)]

We note especially that essential circumstance that the enumerated above properties of typical diffeomorphisms are established only in C^1-topology in the space of C^1-diffeomorphisms. Moreover, the pointed out properties can become nontypical if to require a more smoothness. So, for example, according to the KAM-theory, elliptic periodic points of C^r-smooth two-dimensional symplectic diffeomorphisms are typically stable at $r \geqslant 5$,[(17)] whereas, by property (2), all periodic elliptic points of typical C^1-diffeomorphisms are unstable.

In the present work we establish the existence of C^r-smooth ($r \geqslant 6$) area-preserving symplectic diffeomorphisms which have, in a bounded domain of the phase space, infinitely many isolated generic elliptic periodic points and, hence, infinitely many elliptic islands. Note that we study two parameter families of maps. The main our constructions under consideration are diffeomorphisms with simplest structurally unstable heteroclinic cycles (Fig. 1). Earlier,[(18, 19)] we considered general type diffeomorphisms with similar cycles for the cases where such diffeomorphisms were contracting

[3] Probably, this result for analytical maps can be deduce from ref. 6 at more detailed consideration.

[4] A periodic point is called 1-elliptic if it has one pair of complex eigenvalues $e^{\pm i\varphi}$ with $\varphi \neq 0, \pi$, and all other its eigenvalues do not belong to the unit circle. For two-dimensional case, the 1-elliptic periodic points are elliptic.

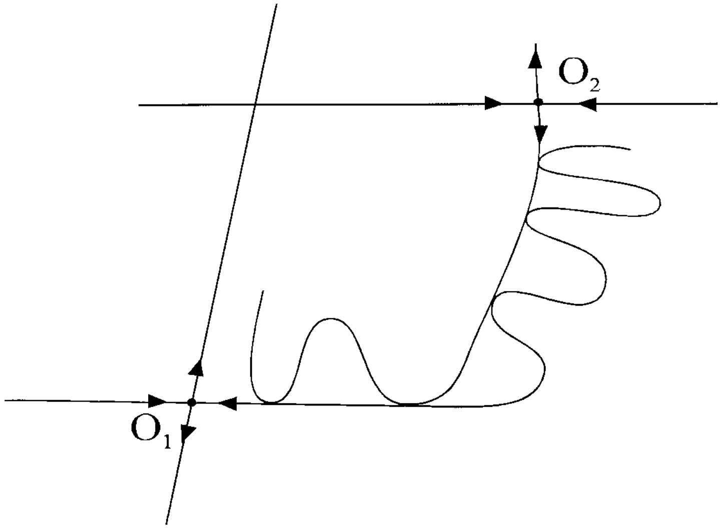

Fig. 1. The example of the simplest structurally unstable heteroclinic cycle.

the area or with "alternating divergence."[5] We have established there conditions of the existence of infinitely many stable periodic points, in the first case, and conditions of the coexistence of infinitely many stable and completely unstable periodic points, in the second case. Note that the closure of a set of such points (in both cases of general and area-preserving diffeomorphisms) contains points of the closure of a set of saddle periodic points. In this connection, we could like to mention also one more perspective direction connected with studying questions of the existence of regions of dense structural instability, so-called Newhouse regions, in the space of symplectic maps or Hamiltonian flows. Recently, the existence of such regions was proved for two-dimensional symplectic maps, namely, for the families of standard maps[(20)] and conservative Henon maps.[(21, 22)]

1. STATEMENT OF THE PROBLEM AND MAIN RESULTS

Let us consider a C^r-smooth ($r \geqslant 6$) two-dimensional area-preserving diffeomorphism T given in some bounded region $G \subset R^2$. Suppose that T has a simplest structurally unstable heteroclinic cycle (as in Fig. 1). We assume that T has two saddle fixed points O_1 and O_2 with eigenvalues λ_i

[5] I.e., in the case where the diffeomorphism contracts the area in a neighbourhood of one of saddle fixed pints belonging to the heteroclinic cycle and expands the area in a neighbourhood of the other point.

and λ_i^{-1} where $|\lambda_i| < 1$, $i = 1, 2$. Also, we assume that the invariant manifolds behave in the following way: $W^u(O_1)$ and $W^s(O_2)$ intersect transversely in the points of a heteroclinic orbit Γ_{12}, and $W^u(O_2)$ and $W^s(O_1)$ have a quadratic tangency along the points of a heteroclinic orbit Γ_{21}. The set $C = \{O_1, O_2, \Gamma_{12}, \Gamma_{21}\}$ is the heteroclinic cycle under consideration.

Let U be a sufficiently small neighbourhood of the heteroclinic cycle: U is a union of small disks U_1 and U_2 containing the points O_1 and O_2 respectively and of a finite number of small disks surrounding those points of orbits Γ_{12} and Γ_{21} which lie outside U_1 and U_2 (Fig. 2).

The maps $T_{0i} \equiv T_{|_{U_i}}$, $i = 1, 2$, are called *local maps*. We will show (see Lemma 1) that in U_i one can to introduce such C^{r-1} (canonical) coordinates (x_i, y_i) that the map T_{0i} is written in the following form

$$\begin{aligned} \bar{x}_i &= \lambda_i x_i (1 + u_1^{(i)} x_i y_i + o(x_i y_i)) \\ \bar{y}_i &= \lambda_i^{-1} y_i (1 - u_1^{(i)} x_i y_i + o(x_i y_i)) \end{aligned} \tag{1.1}$$

where coefficient $u_1^{(i)}$ is an invariant of C^p-smooth ($p \geqslant 3$) canonical changes of variables. Clearly, the equations of manifolds $W^s_{\text{loc}}(O_i)$ and $W^u_{\text{loc}}(O_i)$ are $y_i = 0$ and $x_i = 0$ respectively.

Choose two pairs of heteroclinic points: $M_1^+(x_1^+, 0) \in \Gamma_{21}$ and $M_1^-(0, y_1^-) \in \Gamma_{12}$ in U_1, and $M_2^+(x_2^+, 0) \in \Gamma_{12}$ and $M_2^-(0, y_2^-) \in \Gamma_{21}$ in U_2.

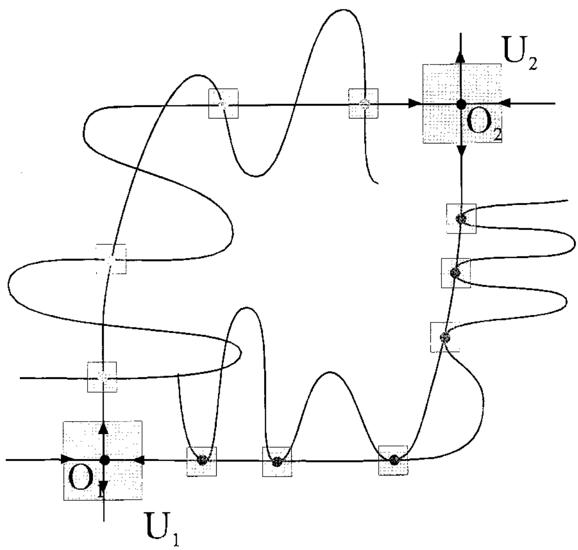

Fig. 2. The neighbourhood of the heteroclinic cycle.

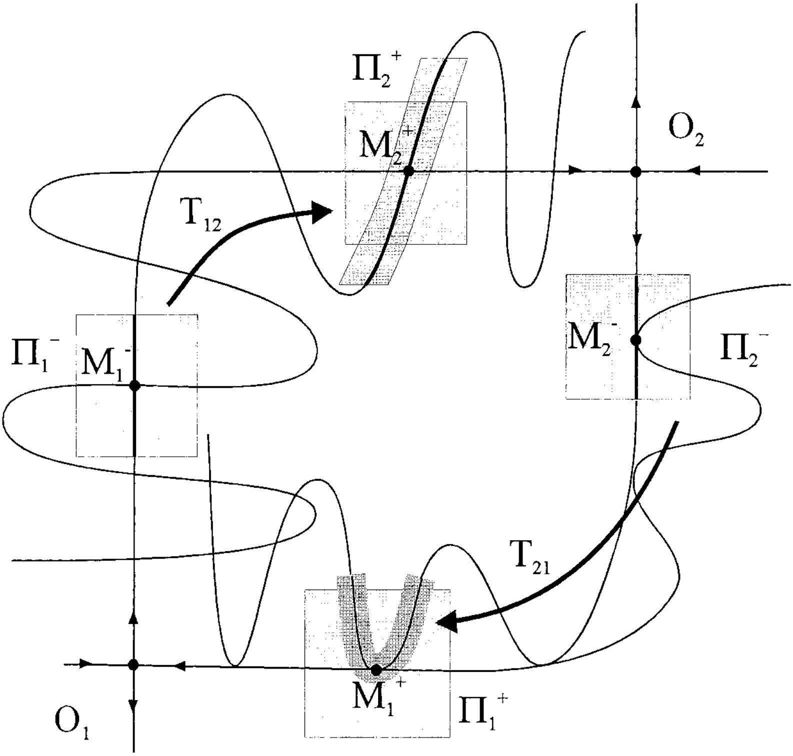

Fig. 3. Rectangle neighbourhoods of heteroclinic points and schematic actions of global maps T_{12} and T_{21}.

Without loss of generality, we assume that $x_2^+ > 0$, $y_1^- > 0$. Let $\Pi_i^+ \subset U_i$ and $\Pi_i^- \subset U_i$ be sufficiently small rectangle neighbourhoods of the points M_i^+ and M_i^- (Fig. 3). We will denote coordinates (x_i, y_i) on Π_i^+ and Π_i^- as (x_{0i}, y_{0i}) and (x_{1i}, y_{1i}), respectively.

Obviously, there are positive integers n_1 and n_2 such that $T^{n_1}(M_1^-) = M_2^+$, $T^{n_2}(M_2^-) = M_1^+$. The corresponding maps $T_{12} \equiv T^{n_1}$ acting from Π_1^- to a small neighborhood of M_2^+ and $T_{21} \equiv T^{n_2}$ acting from Π_2^- to a small neighborhood of M_1^+ are called *global maps*, they are defined by the orbits close to a piece of the heteroclinic orbit Γ_{12} and a piece of the heteroclinic orbit Γ_{21} respectively.

The map T_{12} can be represented as follows

$$\begin{aligned} \bar{x}_{02} - x_2^+ &= a_{12}x_{11} + b_{12}(y_{11} - y_1^-) + O[(|x_{11}| + |y_{11} - y_1^-|)^2] \\ \bar{y}_{02} &= c_{12}x_{11} + d_{12}(y_{11} - y_1^-) + O[(|x_{11}| + |y_{11} - y_1^-|)^2] \end{aligned} \tag{1.2}$$

where $d_{12} \neq 0$ since $W^u(O_1)$ and $W^s(O_2)$ intersect transversally at the point M_2^+.

The map T_{21} can be written in the form

$$\begin{aligned} \bar{x}_{01} - x_1^+ &= a_{21} x_{12} + b_{21}(y_{12} - y_2^-) + O[(|x_{12}| + |y_{12} - y_2^-|)^2] \\ \bar{y}_{01} &= c_{21} x_{21} + d_{21}(y_{12} - y_2^-)^2 \\ &\quad + O[(x_{12}^2 + |x_{12}|\,|y_{12} - y_2^-| + |y_{12} - y_2^-|^3] \end{aligned} \tag{1.3}$$

where $d_{21} \neq 0$ because $W^u(O_2)$ and $W^s(O_1)$ have a quadratic tangency at the point M_1^+; and $b_{21} c_{21} = -1$ since T_{21} preserves the area.

Diffeomorphisms, which possesses a pair of saddle fixed points close to O_1 and O_2 and two heteroclinic orbits (transverse one and the other corresponding to quadratic tangency of the invariant manifolds) close to Γ_{12} and Γ_{21}, respectively, form a locally connected codimension one bifurcation surface H in the space of C^r-smooth area-preserving diffeomorphisms equipped with C^r-topology. We will study, for diffeomorphisms from H, the structure of the set N of orbits entirely lying in U. Analogously to papers refs. 18 and 19, where general diffeomorphisms were considered, one can be established that area-preserving diffeomorphisms with structurally unstable heteroclinic cycles are divided into three classes by the type of description of the structure of N.

If $\lambda_1 > 0$, $\lambda_2 > 0$, $c_{21} < 0$, $d_{21} < 0$ we will say that diffeomorphism T under consideration belongs to *the first class*. (An example of such a diffeomorphism is shown in Fig. 4a). In this case the structure of N is trivial, namely, $N = O_1 \cup O_2 \cup \Gamma_{12} \cup \Gamma_{21}$.

Diffeomorphism T under consideration with $\lambda_1 > 0$, $\lambda_2 > 0$, $c_{21} < 0$, $d_{21} > 0$ belongs to *the second class* (an example is in Fig. 4b). For such diffeomorphisms the set N admits a complete description in terms of the symbolic dynamics and, furthermore, all orbits of N, expect for Γ_{21}, are hyperbolic. It is important also that the structure of N is not changed at (small) perturbations of T inside H.

We group the diffeomorphisms corresponding to the other (different from $\lambda_1 > 0$, $\lambda_2 > 0$, $c_{21} < 0$) combinations of signs of λ_1, λ_2, c_{21} and d_{21} to *the third class* (these are, for instance, the diffeomorphisms with the cycles shown in Figs. 4c, d). Note, that, in the case of diffeomorphisms of the third class, the set N contains nontrivial hyperbolic subsets which, in general, do not coincide with $N \backslash \Gamma_{21}$.[18, 19]

Diffeomorphisms of the third class can be subdivided into several types each of which corresponds to a specific combination of the signs of the quantities λ_1, λ_2, c_{21} and d_{21}. In the cases where λ_1 or λ_2 are negative the signs of c_{21} and d_{21} may change depending on the choice of the points

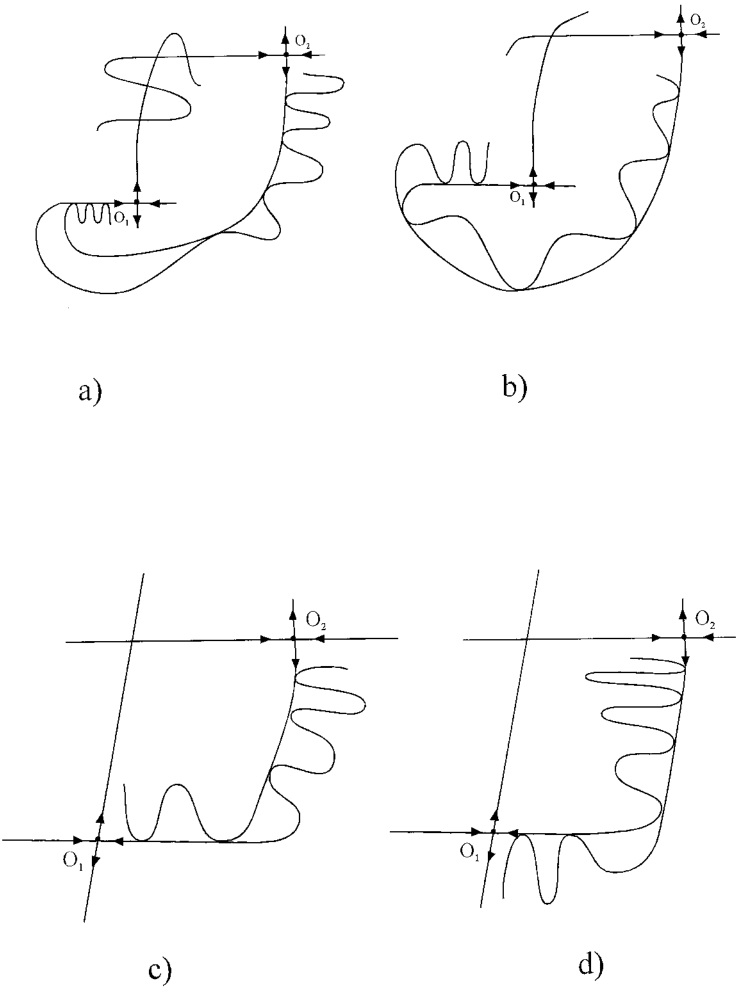

Fig. 4. The examples of diffeomorphisms with structurally unstable heteroclinic cycles: (a) of the first class; (b) of the second class and (c), (d) of the third class.

M_1^- and M_2^+: we may assume in this case, without loss of generality, that $c_{21} > 0$ when λ_2 is negative and $d_{21} > 0$ when λ_1 is negative. As a consequence, we have 7 possible combinations of signs of quantities λ_1, λ_2, c_{21} and d_{21} corresponding to the diffeomorphisms of the third class (Table 1). The set H of diffeomorphisms of the third class will be labeled as H_3. To specify to which of the seven types belong the diffeomorphisms on H_3 we will use the notation H_3^α, $\alpha = 1,..., 7$.

Table 1

	H_3^1	H_3^2	H_3^3	H_3^4	H_3^5	H_3^6	H_3^7
λ_1	+	+	+	+	−	−	−
λ_2	+	+	−	−	+	+	−
c_{21}	+	+	+	+	+	−	+
d_{21}	+	−	+	−	+	+	+

The main attention will be concentrated in studying diffeomorphisms of the third class since both elliptic and parabolic periodic points may exist at such diffeomorphisms.

Recall that a periodic point of diffeomorphism T is called *the elliptic point* of period p if roots of the characteristic equations (i.e., eigenvalues of the matrix of the differential $D(T^p)$) lie on the unit circle and are complex conjugate. If, to addition, the map T^p in some neighbourhood of the elliptic point is reduced to the following complex form

$$\bar{z} = e^{i\psi} z + i e^{i\psi} h(\psi)\, z^2 z^* + O(|z|^4) \tag{1.4}$$

where $h(\psi) \neq 0$, $\psi \neq 2\pi/3$, $\psi \neq \pi/2$, then, as it follows from the KAM-theory, such a point is *the elliptic point of stable type*. It is called also *generic elliptic point*.

A periodic point is called *parabolic* if both roots of the characteristic equations are equal either to $+1$ or to -1. At general conditions, such a point is unstable in the first case, and it may be both stable and unstable in the second case.[4]

Diffeomorphisms with heteroclinic tangencies are remarkable because they possess moduli (i.e., continuous invariants of the topological conjugacy). In this case, two diffeomorphisms with different values of the moduli are not conjugate and, as a consequence, a continuum of classes of topological conjugacy exists. One of the moduli is the quantity

$$\theta = \frac{\ln |\lambda_2|}{\ln |\lambda_1|} \tag{1.5}$$

discovered by Palis.[23] In the case of area-preserving diffeomorphisms of the third class, the quantity θ is an Ω-modulus also. More exactly, θ is a modulus of topological conjugacy on the set of non-wandering orbits entirely lying in a small neighbourhood of the heteroclinic cycle C. Diffeomorphisms of the third class have one more Ω-modulus[18]

$$\tau_0 = [\tau - (n_1 + n_2)](\mathrm{mod}(1+\theta))$$

where

$$\tau = -\frac{1}{\ln|\lambda_1|} \ln \frac{|c_{21} x_2^+|}{|y_1^-|} \tag{1.6}$$

Note, that τ_0 depends also on coefficients of the global maps T_{12} and T_{21}. Nevertheless, τ_0 is an invariant of T (in that sense that τ_0 does not depend on both choosing pairs of heteroclinic points (of the orbits Γ_{12} and Γ_{21}) and canonical coordinate transformations that preserve form (1.1) of the local map T_{0i}).

Note also, that in the case of general two-dimensional diffeomorphisms with a homoclinic tangency the following quantity (introduced in ref. 24)

$$\theta_0 = -\frac{\ln|\lambda|}{\ln|\gamma|}$$

is a modulus, where λ and γ, $|\lambda| < 1$, $|\gamma| > 1$, are multipliers of a periodic orbit whose invariant manifolds have the tangency. But $\theta_0 \equiv 1$ for area-preserving diffeomorphisms with a homoclinic tangency since $\gamma \equiv \lambda^{-1}$. Moreover, such diffeomorphisms (with the tangency) can possess elliptic (parabolic) periodic points only in the case where some quantity $\tilde{\tau}$ (analogous to τ) will be close to an integer (see refs. 25 and 26 for more details).

It follows from definition of Ω-modulus that arbitrary change of its value leads to bifurcations of nonwandering orbits (in particular, periodic and homoclinic ones). First of all, we will interest in bifurcations of so-called *single-round periodic orbits*. Recall, that fixed points of the following maps

$$T_{ij} \equiv T^{n_2+j+n_1+i} : \Pi_1^+ \to \Pi_1^+ \tag{1.7}$$

defined in one round along the neighbourhood of the heteroclinic cycle C, corresponds to such orbits. (In other words, the maps T_{ij} are the Poincaré maps (or the first return maps) for single-round periodic orbits).[6]

[6] The map T_{ij} is represented as the following superposition of the local and global maps

$$T_{ij} \equiv T_{21} T_{02}^j T_{12} T_{01}^i : \Pi_1^+ \to \Pi_1^+$$

where powers i and j of the local maps T_{01} and T_{02} can be how much large here.

We consider the map T_{ij} at sufficiently large i and j. Let θ and τ be such that the following inequality takes place

$$v_{ij}^1 < i - j\theta + \tau - j\lambda_2^j \frac{x_2^+ y_2^-}{\ln|\lambda_1|}\left(u_1^{(2)} - \theta u_1^{(1)} \frac{c_{21} x_1^+}{y_2^-}\right) < v_{ij}^2 \tag{1.8}$$

where quantities $v_{ij}^{1,2}$ are of the order $O(\lambda_1^{2i} + \lambda_2^{2j})$.

Theorem 1. Let θ and τ satisfy inequality (1.8). Then the map T_{ij} has a generic elliptic fixed point.

Note, that inequality (1.8) has infinitely many integer solutions (with respect to i and j) for every θ and τ from a set of points (θ, τ) which is dense in the half-plane $\theta > 0$. Such θ and τ are connected by strong arithmetical relations: they admit so-called "abnormally good" nonhomogeneous approximations (exponential ones) by rational numbers. In other words, for such θ and τ, the straight line $i = j\theta - \tau$ approach abnormally (exponentially) close to points of the integer-value lattice: here, the minimum of the distance has the order $O(j\lambda_2^j)$, when it is "normally" if this minimum is of the order $O(j^{-1})$). We will call *abnormally well approximated* those θ and τ for which inequality (1.8) has infinitely many integer solutions.

Theorem 2. If θ and τ are abnormally well approximated, then diffeomorphism T has infinitely many single-round generic elliptic periodic points (each of which lies inside the own elliptic island). Moreover, the closure of these points contains the heteroclinic cycle C.

This statement[7] admits also the following additional refinement:

If the quantity $Q \equiv u_1^{(2)} - \theta u_1^{(1)}(c_{21} x_1^+ / y_2^-)$ is not equal to zero, then values of θ in hypothesis of Theorem 2 are irrational.

Note, that the quantity Q is an invariant in that sense that it does not depend on a choose of pairs of heteroclinic points of the orbits Γ_{12} and Γ_{21} (see Lemma 5 below).

Consequence. Diffeomorphisms with infinitely many elliptic islands are dense in H_3.

[7] Note, that analogous result was established in refs. 25, 26, and 27 for the case of dissipative systems with homoclinic tangencies. It was proved there that if main Ω-moduli are abnormally well approximating, then the corresponding system possesses infinitely many asymptotically stable periodic orbits (sinks).

Since θ is an Ω-modulus of diffeomorphisms of the third class, it is naturally to consider θ as a control parameter at studying bifurcations in the set of diffeomorphisms from H_3. The following theorem takes place

Theorem 3. Let T_θ be a one-parameter family of diffeomorphisms in H_3 depending smoothly on the parameter θ. Then, in any interval of variation of θ, values of the parameter are dense at which diffeomorphism T_θ has a single-round parabolic periodic point. Such points are both types: with eigenvalues $\nu_1 = \nu_2 = +1$ and $\nu_1 = \nu_2 = -1$. Furthermore, the point is unstable in the first case and stable in the second case.

2. PROPERTIES OF THE LOCAL MAPS

Suppose that a C^r-smooth two-dimensional area-preserving map F_0 has a saddle fixed point O with eigenvalues λ and λ^{-1} where $|\lambda| < 1$. Let F_ε be a parameter family which is C^r-smooth in both variables and parameters and $F_{\varepsilon}|_{\varepsilon=0} \equiv F_0$. One can assume that, for all sufficiently small ε, the fixed point O_ε is in the origin and that the coordinates, x and y, are such that the axes x and y correspond to the proper subspaces for $\lambda(\varepsilon)$ and $\lambda(\varepsilon)^{-1}$, respectively. In this case the map $T_0(\varepsilon) \equiv F_{\varepsilon}|_{U_0}$, where U_0 is a small neighbourhood of the point O_ε, can be written in the form

$$\bar{x} = \lambda(\varepsilon)\, x + \varphi(x, y, \varepsilon), \qquad \bar{y} = \lambda(\varepsilon)^{-1}\, y + \psi(x, y, \varepsilon) \tag{2.1}$$

where functions φ and ψ and their first derivatives in coordinates vanish at $x = y = 0$ for all small ε. In this case the equations of the local stable and local unstable manifolds can be written as $y = h_s(x, \varepsilon)$ and $x = h_u(y, \varepsilon)$, respectively, where h_s and h_u are C^r-smooth and such that

$$h_s(0, \varepsilon) = \frac{\partial h_s(0, \varepsilon)}{\partial x} = 0, \qquad h_u(0, \varepsilon) = \frac{\partial h_u(0, \varepsilon)}{\partial y} = 0$$

If to make two consecutive changes of variables of the form

$$\xi = x - h_u(y, \varepsilon), \qquad \eta = y \text{ and } \xi = x, \qquad \eta = y - h_s(x, \varepsilon) \tag{2.2}$$

each of which is C^r-smooth and area-preserving, then the map $T_0(\varepsilon)$ is brought to the following form (we retain the old coordinate notation):

$$\bar{x} = \lambda(\varepsilon)\, x + f(x, y, \varepsilon)\, x, \qquad \bar{y} = \lambda(\varepsilon)^{-1}\, y + g(x, y, \varepsilon)\, y \tag{2.3}$$

where $f(0, 0, \varepsilon) = 0$, $g(0, 0, \varepsilon) = 0$. Form (2.3) corresponds to the case where both the local stable and local unstable invariant manifolds of the point O_ε

are straightened: the equation of $W^s_{\mathrm{loc}}(O_\varepsilon)$ and $W^u_{\mathrm{loc}}(O_\varepsilon)$ are $y=0$ and $x=0$, respectively (for all sufficiently small ε). Form (2.3) of the map $T_0(\varepsilon)$ is more convenient than (2.1) but its using gives some technical difficulties. This is connected, in particular, with the fact that "too much" nonresonant terms are in the right-hand side of (2.3). Thus, the question is very important on a reduction of map (2.3) to a more simple form by means sufficiently smooth and area-preserving changes of coordinates.

It is clear that the simplest form is the linear form of $T_0(\varepsilon)$. But only C^1-linearization is possible here.[(28)] On the other hand, for the analytical case, J. Moser[(29)] has established that the map T_0 can be reduced to the following normal form

$$\bar{x}=\lambda B(xy)\,x, \qquad \bar{y}=\lambda^{-1}B^{-1}(xy)\,y \tag{2.4}$$

where

$$\begin{aligned} B(xy) &\equiv 1+\beta_1\cdot xy+\beta_2\cdot(xy)^2+\cdots+\beta_n\cdot(xy)^n+\cdots \\ B^{-1}(xy) &\equiv 1+\tilde{\beta}_1\cdot xy+\tilde{\beta}_2\cdot(xy)^2+\cdots+\tilde{\beta}_n\cdot(xy)^n+\cdots \end{aligned} \tag{2.5}$$

are serieses in monomials (xy) converging in some neighbourhood of the origin. Since the Jacobian of (2.4) is equal to one identically, it follows that coefficients β_i and $\tilde{\beta}_i$ are connected by some relations. For example, $\beta_1=-\tilde{\beta}_1$, $\tilde{\beta}_2=\beta_1^2-\beta_2$ etc.

In the general smooth case ($r\leqslant\infty$) the map $T_0(\varepsilon)$ can be brought to some finitely smooth "normal forms" which, in some sense, are similar to (2.4). For our goal it will be sufficient the first order "normal form" whose existence is proved in the following lemma

Lemma 1. For all sufficiently small ε there exists a canonical C^{r-1}-change of variables (C^{r-2}-smooth in parameters) bringing $T_0(\varepsilon)$ to the form

$$\begin{aligned} \bar{x} &= \lambda(\varepsilon)\,x(1+\beta_1(\varepsilon)\,xy+O[x^2\,|y|+|x|\,y^2]) \\ \bar{y} &= \lambda^{-1}(\varepsilon)\,y(1-\beta_1(\varepsilon)\,xy+O[x^2\,|y|+|x|\,y^2]) \end{aligned} \tag{2.6}$$

Proof. By (2.3), the C^r-map $T_0(\varepsilon)$ at $r\geqslant 3$ can be written in the following "extended form"

$$\begin{aligned} \bar{x} &= \lambda(\varepsilon)\,x+\varphi_1(x,\varepsilon)+\psi_1(y,\varepsilon)\,x+f_1(x,y,\varepsilon)\,x^2y \\ \bar{y} &= \lambda^{-1}(\varepsilon)\,y+\varphi_2(x,\varepsilon)\,y+\psi_2(y,\varepsilon)+g_1(x,y,\varepsilon)\,xy^2 \end{aligned} \tag{2.7}$$

where $\varphi_1, \psi_2 \in C^r$, $\varphi_1(0, \varepsilon) = \partial\varphi_1(0, \varepsilon)/\partial x \equiv 0$, $\psi_2(0, \varepsilon) = \partial\psi_2(0, \varepsilon)/\partial y \equiv 0$, $\varphi_2(0, \varepsilon) = \psi_1(0, \varepsilon) \equiv 0$. Lemma 1 states that $T_0(\varepsilon)$ can be brought to the form where $\varphi_i \equiv 0$, $\psi_i \equiv 0$, $i = 1, 2$. Note, that functions φ_i and ψ_i are non-resonant (i.e., they do not contain resonant monomials).

We will use canonical changes of variables. Recall, that the *canonical change* is constructed by means of *the generating function* as follows. Let $V(x, \eta, \varepsilon)$ be a sufficiently smooth function of variables x, η and parameters ε and such that $V(0, 0, 0) = 0$ and $V_{x\eta}(0, 0, 0) \neq 0$. Then, the canonical change of variables with given generating function V is the change $(x, y) \to (\xi, \eta)$ of the form

$$\xi = \frac{\partial V(x, \eta, \varepsilon)}{\partial \eta}, \qquad y = \frac{\partial V(x, \eta, \varepsilon)}{\partial x}$$

If V is sufficiently smooth, then, for small x, η and ε, the canonical change is a diffeomorphism and preserves the area. Note to the point, that changes (2.2) are both canonical with generating functions

$$x\eta + \int_0^\eta h_u(t, \varepsilon)\, dt \qquad \text{and} \qquad x\eta - \int_0^x h_s(t, \varepsilon)\, dt$$

respectively.[8]

For proving the lemma we will make two consecutive canonical changes with generating functions of the following form

$$V_1 = x\eta + v_1(x, \varepsilon)\, \eta \qquad \text{and} \qquad V_2 = x\eta + v_2(\eta, \varepsilon)\, x$$

where $v_1(0, \varepsilon) = \partial v_1(0, \varepsilon)/\partial x \equiv 0$, $v_2(0, \varepsilon) = \partial v_2(0, \varepsilon)/\partial \eta \equiv 0$. As we will show, the first and second changes (with appropriate functions v_1 and v_2) annihilate functions φ_1 and ψ_2 in (2.7), respectively; after that, as we will show, functions φ_2 and ψ_1 vanish automatically because symplecticity.

The first change is

$$\xi = \frac{\partial V_1(x, \eta, \varepsilon)}{\partial \eta} \equiv x + v_1(x, \varepsilon), \qquad y = \frac{\partial V_1(x, \eta, \varepsilon)}{\partial x} \equiv \eta + \tilde{v}_1(x, \varepsilon)\, \eta \tag{2.8}$$

where $\tilde{v}_1 \equiv \partial v_1(x, \varepsilon)/\partial x$ and $\tilde{v}_1(0, \varepsilon) \equiv 0$.

[8] Note, that these changes are C^r-smooth in variables but C^{r-1} in parameters. This is one of causes that the smoothness of next changes in parameters are less than the smoothness in variables.

The first equation of (2.7) is transformed as $\bar{\xi} = \bar{x} + v_1(\bar{x}, \varepsilon)$. Note that

$$\begin{aligned} v_1(\bar{x}, \varepsilon) &= v_1(\lambda x + \varphi_1(x, \varepsilon), \varepsilon) + [v_1(\lambda x + \varphi_1(x, \varepsilon) + \psi_1(y, \varepsilon)\, x + f_1 x^2 y, \varepsilon) \\ &\quad - v_1(\lambda x + \varphi_1(x, \varepsilon), \varepsilon)] \\ &= v_1(\lambda x + \varphi_1(x, \varepsilon), \varepsilon) + O(x^2 y) \end{aligned}$$

since $v_1(0, \varepsilon) = \partial v_1(0, \varepsilon)/\partial x \equiv 0$. Also

$$\begin{aligned} \psi_1(y, \varepsilon)\, x &\Rightarrow \psi_1(\eta + \tilde{v}_1(x, \varepsilon)\, \eta, \varepsilon)\, x \\ &= \psi_1(\eta, \varepsilon)\, x + [\psi_1(\eta + \tilde{v}_1(x, \varepsilon)\, \eta, \varepsilon) - \psi_1(\eta, \varepsilon)]\, x \\ &\Rightarrow \psi_1(\eta, \varepsilon)\, \xi + O(\xi^2 \eta) \end{aligned}$$

since $\tilde{v}_1(0, \varepsilon) \equiv 0$. Then, one has

$$\begin{aligned} \bar{\xi} &= \lambda(\varepsilon)(\xi - v_1(x, \varepsilon)) + \varphi_1(x, \varepsilon) + \psi_1(y, \varepsilon)\, x \\ &\quad + v_1(\lambda(\varepsilon)\, x + \varphi_1(x, \varepsilon) + \psi_1(y, \varepsilon)\, x + f_1 x^2 y, \varepsilon) + \cdots \\ &= \lambda(\varepsilon)\, \xi + [\varphi_1(x, \varepsilon) - \lambda(\varepsilon)\, v_1(x, \varepsilon) + v_1(\lambda(\varepsilon)\, x + \varphi_1(x, \varepsilon), \varepsilon))] \\ &\quad + \psi_1(\eta, \varepsilon)\, \xi + \cdots \end{aligned} \tag{2.9}$$

where the ellipsis denote terms of the order $O(\xi^2 \eta)$.

Assume now that the expression in the square brackets in (2.9) is equal to zero, i.e., $v_1(x, \varepsilon)$ satisfies the following equation

$$v_1[\lambda(\varepsilon)\, x + \varphi_1(x, \varepsilon), \varepsilon] = \lambda(\varepsilon)\, v_1(x, \varepsilon) - \varphi_1(x, \varepsilon) \tag{2.10}$$

Let us show that Eq. (2.10) has a solution in the class of C^r-functions. Note for this, that (2.10) can be considered as some functional equation which defines an invariant curve of the form $u = v(x, \varepsilon)$ for the following map of the plane:

$$\bar{u} = \lambda(\varepsilon)\, u - \varphi_1(x, \varepsilon), \qquad \bar{x} = \lambda(\varepsilon)\, x + \varphi_1(x, \varepsilon) \tag{2.11}$$

where $\varphi_1(0, \varepsilon) = \partial \varphi_1(0, \varepsilon)/\partial x = 0$. Show that map (2.11) has such an invariant curve. Consider the variables $z = u + x$. Then, (2.11) can be rewritten as

$$\bar{z} = \lambda(\varepsilon)\, z, \qquad \bar{x} = \lambda(\varepsilon)\, x + \varphi_1(x, \varepsilon) \tag{2.12}$$

Since $|\lambda| \neq 1$, $\varphi_1 \in C^r$ and $r \geqslant 3$, this map (as one-dimensional in x) admits a C^r-linearization[28] for all sufficiently small x and ε. The corresponding linear map, $\bar{z} = \lambda(\varepsilon)\, z$, $\bar{x} = \lambda(\varepsilon)\, x$, has a fixed point, which is so-called a "diacritic node." In this case, every straight line, passing through this point, is invariant. Thus, map (2.12) has a C^r-smooth invariant curve, $z = x + v_1(x, \varepsilon)$, that touches the line $z = x$. As a consequence, map (2.11) has a C^r-smooth invariant curve whose equation is $u = v_1(x, \varepsilon)$ where $v_1(0, \varepsilon) = \partial v_1(0, \varepsilon)/\partial x = 0$, as was to be proved.

Thus, after change (2.8) the first equation of (2.7) is transformed as

$$\bar{\xi} = \lambda(\varepsilon)\,\xi + \psi_1(\eta, \varepsilon)\,\xi + O(\xi^2\eta) \tag{2.13}$$

(but functions in the right side of (2.13) is C^{r-1}-smooth, in general, since change (2.8) is C^{r-1}). The second equation of (2.7) after change (2.8) takes the form $\bar{\eta} = \bar{y}(1 + \hat{v}_1(\bar{x}, \varepsilon))$ where

$$\hat{v}_1 \equiv -\frac{\tilde{v}}{1 + \tilde{v}} \qquad \text{and} \qquad \hat{v}_1(0, \varepsilon) \equiv 0$$

Thus,

$$\begin{aligned} \bar{\eta} &= (\lambda(\varepsilon)^{-1}\, y + \psi_2(y, \varepsilon) + \varphi_2(x, \varepsilon)\, y)(1 + \hat{v}_1(\bar{x}, \varepsilon)) + O(\xi\eta^2) \\ &= \lambda(\varepsilon)^{-1}\, \eta + \psi_2(\eta, \varepsilon) + \hat{\varphi}_2(\xi, \varepsilon)\, \eta + O(\xi\eta^2) \end{aligned}$$

since

$$\begin{aligned} \psi_2(\eta(1 + \hat{v}_1(x, \varepsilon)), \varepsilon) &= \psi_2(\eta, \varepsilon) + [\psi_2(\eta(1 + \hat{v}_1(x, \varepsilon)), \varepsilon) - \psi_2(\eta, \varepsilon)] \\ &= \psi_2(\eta, \varepsilon) + O(\xi\eta) \end{aligned}$$

So, after canonical C^{r-1}-change (2.8), map (2.7) takes the form (for the old coordinate notations)

$$\begin{aligned} \bar{x} &= \lambda(\varepsilon)\, x + \psi_1(y, \varepsilon)\, x + f_1^1(x, y, \varepsilon)\, x^2 y \\ \bar{y} &= \lambda(\varepsilon)^{-1}\, y + \psi_2(y, \varepsilon) + \hat{\varphi}_2(x, \varepsilon)\, y + g_1^1(x, y, \varepsilon)\, x y^2 \end{aligned} \tag{2.14}$$

where $\psi_2 \in C^r$, $\psi_2(0, \varepsilon) = \partial\psi_{2y}(0, \varepsilon)/\partial y \equiv 0$, $\psi_1(0, \varepsilon) \equiv 0$, $\hat{\varphi}_2(0, \varepsilon) \equiv 0$.

Conduct now the second canonical change of coordinates (i.e., with the generating function $V_2 = x\eta + v_2(\eta, \varepsilon)\, x$). This change has the form

$$\xi = x + \tilde{v}_2(\eta, \varepsilon)\, x, \qquad y = \eta + v_2(\eta, \varepsilon) \tag{2.15}$$

where $\tilde{v}_2 = \partial v_2/\partial \eta$. Since $v_2(0, \varepsilon) = \partial \tilde{v}_2(0, \varepsilon)/\partial \eta = 0$, the second equation of (2.15), by the implicit function theorem, can be solved with respect to η: $\eta = y + r_2(y, \varepsilon)$. Furthermore, $r_2(y, \varepsilon) + v_2(r_2(y, \varepsilon), \varepsilon) \equiv 0$ and $r_2(0, \varepsilon) = \partial r_2(0, \varepsilon)/\partial y = 0$. Thus, change (2.15) can be represented as

$$\xi = x + \tilde{v}_2(\eta, \varepsilon)\, x, \qquad \eta = y + r_2(y, \varepsilon)$$

Note, that this change is of the same type as (2.8). Therefore, in the same way, it is shown that the second equation of (2.14) is transformed as follows

$$\bar{\eta} = \lambda(\varepsilon)^{-1}\,\eta + \hat{\varphi}_2(\xi, \varepsilon)\,\eta + O(\xi\eta^2) \tag{2.16}$$

Finally, we obtain that, after two consecutive canonical changes (2.8) and (2.15), map (2.14) takes the following form (in the old coordinate notations)

$$\begin{aligned} \bar{x} &= \lambda(\varepsilon)\,x + \hat{\psi}_1(y, \varepsilon)\,x + \tilde{f}_1(x, y, \varepsilon)\,x^2 y \\ \bar{y} &= \lambda(\varepsilon)^{-1}\,y + \hat{\varphi}_2(x, \varepsilon)\,y + \tilde{g}_1(x, y, \varepsilon)\,xy^2 \end{aligned} \tag{2.17}$$

where the right side is C^{r-1}, in general, and $\hat{\psi}_1(0, \varepsilon) \equiv 0$, $\hat{\varphi}_2(0, \varepsilon) \equiv 0$.

Since the map (2.17) preserves the area, its Jacobian is equal to one identically, i.e.,

$$J(x, y, \varepsilon) = 1 + \lambda(\varepsilon)^{-1}\,\hat{\psi}_1(y, \varepsilon) + \lambda(\varepsilon)\,\hat{\varphi}_2(x, \varepsilon) + O(xy) \equiv 1$$

One has $J(0, y, \varepsilon) = 1 + \lambda(\varepsilon)^{-1}\,\hat{\psi}_1(y, \varepsilon) \equiv 1$. It implies that $\hat{\psi}_1(y, \varepsilon) \equiv 0$. Further, $J(x, 0, \varepsilon) = 1 + \lambda(\varepsilon)\,\hat{\varphi}_2(x, \varepsilon) \equiv 1$, that is $\hat{\varphi}_2(x, \varepsilon) \equiv 0$. This completes the proof of the lemma.

Coordinates of Lemma 1 are very convenient since iterations of the map T_0, which is given in form (2.6), are asymptotically close to iterations of the map T_0 in the case where it is represented in the Birkhoff–Moser normal form (2.4). Namely, the following lemma takes place

Lemma 2. If $T_0(\varepsilon)$ has form (2.6), then $T_0^k(\varepsilon)$ can be represented as follows

$$\begin{aligned} x_k &= \lambda(\varepsilon)^k\,x_0(1 + k\lambda(\varepsilon)^k\,\beta_1(\varepsilon)\,x_0 y_k) + O(\lambda(\varepsilon)^{2k}) \\ y_0 &= \lambda(\varepsilon)^k\,y_k(1 + k\lambda(\varepsilon)^k\,\beta_1(\varepsilon)\,x_0 y_k) + O(\lambda(\varepsilon)^{2k}) \end{aligned} \tag{2.18}$$

where the functions in the right-hand side of (2.18) are C^{r-1}-smooth and uniformly bounded in k with all derivatives up to order $(r-2)$.

Proof. The proof is based on the method of boundary value problem.[30, 32] By Lemma 1, the map $T_0(\varepsilon)$ (C^{r-1}-smooth) can be written in the form

$$\begin{aligned} \bar{x} &= \lambda(\varepsilon)\,x + \tilde{f}(x, y, \varepsilon) \equiv \lambda(\varepsilon)\,x(1 + \beta_1(\varepsilon)\,xy) + O(x^2y^2 + |x^3y|) \\ \bar{y} &= \lambda(\varepsilon)^{-1}\,y + \tilde{g}(x, y, \varepsilon) \equiv \lambda(\varepsilon)^{-1}\,y(1 - \beta_1(\varepsilon)\,xy) + O(x^2y^2 + |xy^3|) \end{aligned} \tag{2.19}$$

Consider the following operator **P**:

$$\bar{x}_j = \lambda(\varepsilon)^j\, x_0 + \sum_{s=0}^{j-1} \lambda(\varepsilon)^{j-s-1}\, \tilde{f}(x_s, y_s, \varepsilon)$$
$$\bar{y}_j = \lambda(\varepsilon)^{k-j}\, y_k - \sum_{s=j}^{k-1} \lambda(\varepsilon)^{s-j+1}\, \tilde{g}(x_s, y_s, \varepsilon) \tag{2.20}$$

(where $j = 0, 1,..., k$) defined on the set

$$R(\delta) = \{z = [(x_j, y_j)]_{j=0}^{k}, \|z\| \leqslant \delta\}$$

where $\|z\|$ is the maximum of the absolute value of components x_j, y_j of vector z and δ is a positive small quantity. If $z_0 = [(x_j^0, y_j^0)]_{j=0}^{k}$ is a fixed point of **P**, then the following diagram takes place

$$(x_0^0, y_0^0) \xrightarrow{T_0} (x_1^0, y_1^0) \xrightarrow{T_0} \cdots \xrightarrow{T_0} (x_k^0, y_k^0)$$

It was proved in ref. 32 that, for all sufficiently small ε and $\delta = \delta_0$ and $\|x_0\| \leqslant \delta_0/2$, $|y_k| \leqslant \delta_0/2$, operator **P** maps region $R(\delta_0)$ into itself and is contracting. Thus, map (2.20) has a unique fixed point $z_0 = [(x_j^0(x_0, y_k), y_j^0(x_0, y_k)]_{j=0}^{k}$. Due to the contractibility, its coordinates x_j^0 and y_j^0 can be found, for example, by the method of successive approximations. As an initial approximation one takes the following solution of the linear problem

$$x_j^{(0)} = \lambda(\varepsilon)^j\, x_0, \qquad y_j^{(0)} = \lambda(\varepsilon)^{k-j}\, y_k$$

In virtue of (2.19) and (2.20), for the first approximation we have such a representation

$$\begin{aligned} x_j^{(1)} &= \lambda^j x_0 + \sum_{s=0}^{j-1} \lambda^{j-s}\{\beta_1 \lambda^{2s} \lambda^{k-s} x_0^2 y_k + O(\lambda^{2s}\lambda^k + \lambda^{2k})\} \\ &= \lambda^j x_0 + \beta_1 j \lambda^{j+k} x_0^2 y_k + O(\lambda^{j+k}) \\ y_j^{(1)} &= \lambda^{k-j} y_k - \sum_{s=j}^{k-1} \lambda^{s-j+1}\{-\lambda^{-1}\beta_1 \lambda^s \lambda^{2k-2s} x_0 y_k^2 + O(\lambda^{2k} + \lambda^s \lambda^{3k-3s})\} \\ &= \lambda^{k-j} y_k + \beta_1 (k-j)\, \lambda^{2k-j} x_0 y_k^2 + O(\lambda^{2k-j}) \end{aligned} \tag{2.21}$$

where $\lambda \equiv \lambda(\varepsilon)$, $\beta_1 \equiv \beta_1(\varepsilon)$. It is clear that next approximations have the same form. Thus, we have the following representation for coordinates of the fixed point of **P**

$$\begin{aligned}
&x_0^0 = x_0 \\
&y_k^0 = y_k \\
&x_j^0 = \lambda^j x_0 (1 + \beta_1 j \lambda^k x_0 y_k) + O(\lambda^{j+k}) \\
&y_j^0 = \lambda^{k-j} y_k (1 + \beta_1 (k-j)\, \lambda^k x_0 y_k) + O(\lambda^{2k-j}), \qquad j = 0, 1,..., k
\end{aligned} \tag{2.22}$$

Formulas (2.18) follow now from (2.22) (the first for $j = k$ and the second for $j = 0$).

Estimates for derivatives of functions x_j^0 and y_j^0 with respect to the boundary conditions x_0 and y_k and parameters ε are found as follows (see ref. 27 for more details). First, we note that functions x_j^0 and y_j^0 has the same smoothness as the map T_0. Indeed, x_j^0 and y_j^0 can be solved with respect to x_0^0 and y_0^0 as jth iterations of smooth map T_0: $x_j^0 = \tilde{x}_j(x_0^0, y_p^0 0)$, $y_j^0 = \tilde{y}_j(x_0^0, y_0^0)$. The equation $y_k^0 = \tilde{y}_k(x_0^0, y_0^0)$ is solved with respect to y_0^0 by the implicit function theorem, since $\|(\partial y_0^0/\partial y_k)^{-1}\|$ is bounded and separated from zero.[32] Thus, $y_0^0 = y'(x_0, y_k) \in C^{r-1}$, in general. Since $\tilde{x}_j, \tilde{y}_j \in C^{r-1}$ too, the functions

$$x_j^0(x_0, y_k) \equiv \tilde{x}_j(x_0, y_k'(x_0, y_k)), \qquad y_j^0(x_0, y_k) \equiv \tilde{y}_j(x_0, y_k'(x_0, y_k))$$

are C^{r-1}-smooth also.

Let us show that for derivatives

$$\Phi^j_{pqv} \equiv \frac{\partial^l x_j^0}{\partial x_0^p\, \partial y_k^q\, \partial \varepsilon^v}, \qquad \Psi^j_{pqv} \equiv \frac{\partial^l y_j^0}{\partial x_0^p\, \partial y_k^q\, \partial \varepsilon^v}, \qquad l = p + q + v \leqslant r - 2$$

the statement of the lemma is also true. The formal differentiation of both sides of operator **P** (p times in x_0, q times in y_k and v times in ε) gives us the following formula

$$\begin{aligned}
\bar{\Phi}^j_{pqv} &= \sum_{s=0}^{j-1} \lambda^{j-s-1} \left(\frac{\partial \tilde{f}}{\partial x_s} \Phi^s_{pqv} + \frac{\partial \tilde{f}}{\partial y_s} \Psi^s_{pqv} \right) + f_j^l \\
\bar{\Psi}^j_{pqv} &= -\sum_{s=j}^{k-1} \lambda^{s-j+1} \left(\frac{\partial \tilde{g}}{\partial x_s} \Phi^s_{pqv} + \frac{\partial \tilde{g}}{\partial y_s} \Psi^s_{pqv} \right) + g_j^l
\end{aligned} \tag{2.23}$$

where the functions f_j^l and g_j^l depend only on derivatives of orders less than l. Under hypothesis that norms of these last derivatives satisfy, estimates of Lemma 2 we can calculate that the norms of functions f_j^l and g_j^l satisfy the same estimates but with new weight coefficients. Now, since operator (2.23) is linear and contracting for $\delta \leqslant \delta_0$, the norms of derivatives of the order l will also satisfy required estimates. By induction in l, we obtain required proposition. This completes the proof of the lemma.

3. CONSTRUCTION OF THE MAPS T_{ij}

We will find here formulas for the first return maps T_{ij} (see formula (1.7) above) for various sufficiently large i and j.

The set of initial points in Π_α^+, $\alpha = 1, 2$, whose orbits get into Π_α^-, consists of infinitely many horizontal strips $\sigma_k^{0\alpha} = \Pi_\alpha^+ \cap T_{0\alpha}^{-k} \Pi_\alpha^-$. These strips accumulate on $W_{\text{loc}}^s(O_\alpha)$ as $k \mapsto \infty$ (Fig. 5a). Images of strips $\sigma_k^{0\alpha}$ with

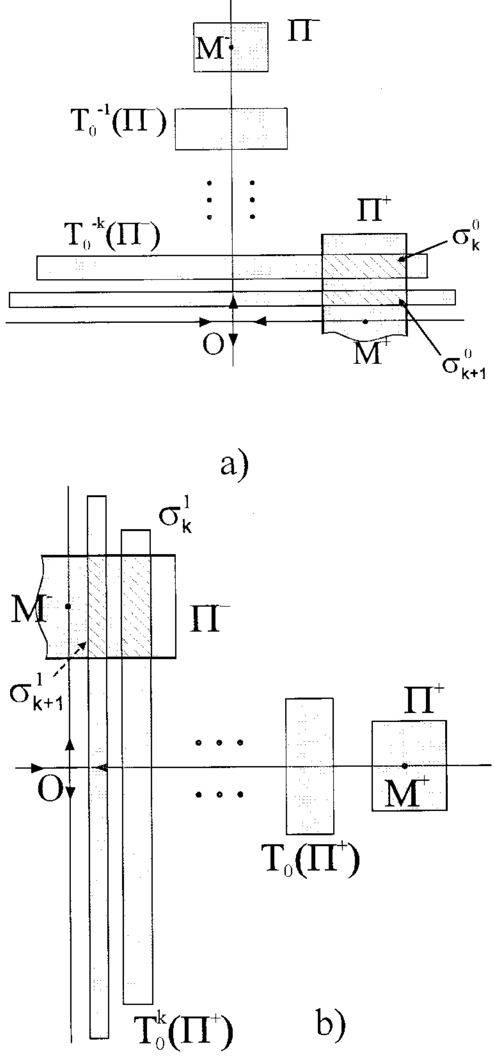

Fig. 5. Schematic actions of local (saddle) maps: (a) at backward iterations and (b) at forward iterations.

respect to the maps $T_{0\alpha}^k$ are vertical strips $\sigma_k^{1\alpha} \equiv T_{0\alpha}^k(\sigma_k^{0\alpha})$ in Π_α^- which accumulate on $W_{\mathrm{loc}}^u(O_\alpha)$ (Fig. 5b). Without loss of generality, we may assume that Π_α^+ and Π_α^- contain entirely the strips $\sigma_k^{0\alpha}$ and $\sigma_k^{1\alpha}$ with numbers $k \geqslant \bar{k}_\alpha$ and $T_{0\alpha}^k \Pi_\alpha^+ \cap \Pi_\alpha^- = \varnothing$ if $k < \bar{k}_\alpha$, where $\bar{k}_1$ and $\bar{k}_2$ are some sufficiently large integers.

By Lemma 2, the map $T_{0\alpha}^k : \sigma_k^{0\alpha} \to \sigma_k^{1\alpha}$ can be represented in the following form:

$$\begin{aligned} \bar{x}_\alpha &= \lambda_\alpha^k x_\alpha (1 + k\lambda_\alpha^k x_\alpha \bar{y}_\alpha u_1^{(\alpha)}) + O(\lambda_\alpha^{2k}) \\ y_\alpha &= \lambda_\alpha^k \bar{y}_\alpha (1 + k\lambda_\alpha^k x_\alpha \bar{y}_\alpha u_1^{(\alpha)}) + O(\lambda_\alpha^{2k}) \end{aligned} \tag{3.1}$$

where $(x_\alpha, y_\alpha) \in \sigma_k^{0\alpha}$, $(\bar{x}_\alpha, \bar{y}_\alpha) \in \sigma_k^{1\alpha}$.

Evidently, orbits from N must intersect the neighbourhood Π_2^+ in intersection points of images $T_{12}(\sigma_m^{11})$ of the strips from Π_1^- with the strips σ_k^{01} for all possible $m \geqslant \bar{k}_1$ and $k \geqslant \bar{k}_2$. Analogously, intersection points of orbits from N with the neighbourhood Π_1^+ must belong to the intersections of images $T_{21}(\sigma_j^{12})$ of the strips from Π_2^- with the strips σ_i^{01} for various $i \geqslant \bar{k}_1$ and $j \geqslant \bar{k}_2$.

Since Γ_{12} is the orbit of transverse intersection of manifolds $W^u(O_1)$ and $W^s(O_2)$, the intersection of any strip $T_{12}\sigma_k^{11}$ with any strip σ_j^{02} consists of one connected component if k and j are sufficiently large (see

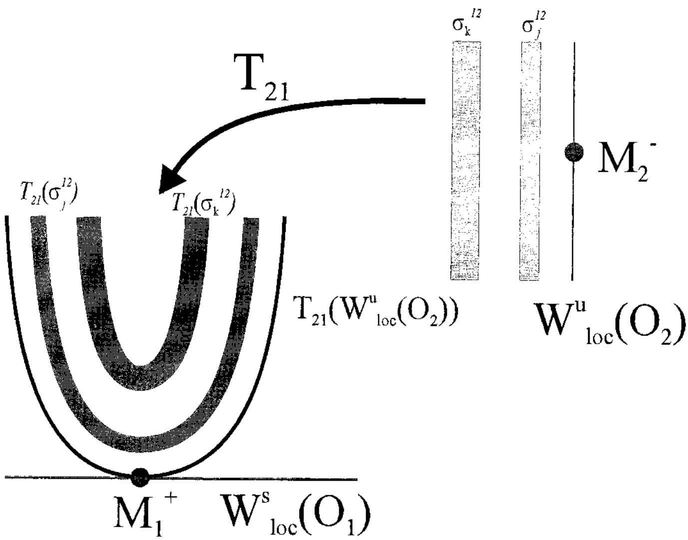

Fig. 6. The images $T_{21}(\sigma_j^{12}) \subset \Pi_1^+$ of the strips $\sigma_j^{12} \subset \Pi_2^-$ have a shape of horseshoes.

(1.2)). It is seen from (1.3) that the images $T_{21}(\sigma_j^{12})$ of the strips σ_j^{12} have a shape of horseshoes accumulated on the "parabola" $T_{21}(W^u_{\mathrm{loc}}(O_2)) \subset W^u(O_2) \cap \Pi_1^+$ (Fig. 6). Orbits of the set N must intersect the neighbourhood Π_1^+ at points of intersection of the horseshoes $T_{21}(\sigma_j^{12})$ and the strips σ_i^{01} for $i \geqslant \bar{k}_1$ and $j \geqslant \bar{k}_2$. Hence, the structure of N depends essentially on the structure of the set of these intersections.

We will speak that the horseshoe $T_{21}(\sigma_j^{12})$ has *a regular intersection* with the strip σ_i^{01}, if the following conditions are fulfilled: (1) the set $T_{21}(\sigma_j^{12}) \cap \sigma_i^{01}$ consists of two connected components, Δ_{ij}^3 and Δ_{ij}^4; (2) the map $T_{21}T_{02}^j$, restricted onto the preimage $(T_{21}T_{02}^j)^{-1}\Delta_{ij}^\alpha \subset \sigma_j^{02}$ of the component Δ_{ij}^α, $\alpha = 3, 4$, is *a saddle map* in the sense of ref. 30 (roughly speaking, this map is expanding with respect to coordinate y_{02} and contracting with respect to coordinate x_{02}).

Different types of intersections of the horseshoe $T_{21}(\sigma_j^{12})$ with the strips lying in Π_1^+ are shown in Fig. 7. The horseshoe has regular intersection with the strip σ_i^{01}, irregular intersection with σ_k^{01} and empty intersection with σ_s^{01}.

The following lemma, which has been proved in ref. 19 (see also ref. 18), gives sufficient conditions for regular and empty intersections of the horseshoes and the strips.

Lemma 3. There are a positive constant S_1 and sufficiently large integers $\bar{k}_1$ and $\bar{k}_2$ such that for any $i \geqslant \bar{k}_1$, $j \geqslant \bar{k}_2$

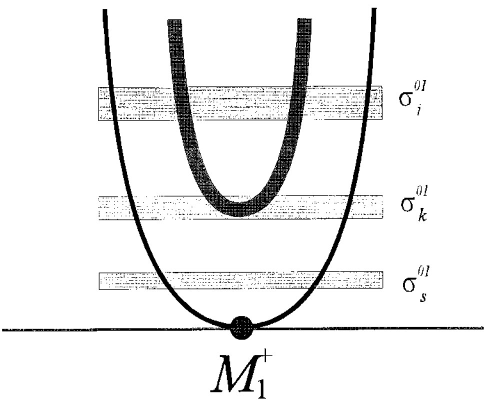

Fig. 7. Types of intersections of the horseshoe $T_{21}(\sigma_j^{12})$ with the strips. The horseshoe has a regular intersection with the strip σ_i^{01}, an irregular intersection $\sigma\sigma_k^{01}$ and empty intersection with σ_s^{01}.

(1) if the inequality

$$d_{21}(\lambda_1^i y_1^- - c_{21}\lambda_2^j x_2^+) > S_{ij}(\bar{k}_1, \bar{k}_2) \tag{3.2}$$

is fulfilled where $S_{ij} = S_1(|\lambda_1|^i + |\lambda_2|^j)(|\lambda_1|^{\bar{k}_1} + |\lambda_2|^{\bar{k}_2} + i\,|\lambda_1|^i + j\,|\lambda_2|^j)$, then the intersection of the horseshoe $T_{21}(\sigma_j^{12})$ with the strip σ_i^{01} is regular;

(2) if the inequality

$$d_{21}(\lambda_1^i y_1^- - c_{21}\lambda_2^j x_2^+) < -S_{ij}(\bar{k}_1, \bar{k}_2) \tag{3.3}$$

is fulfilled, then $T_{21}(\sigma_j^{12}) \cap \sigma_i^{01} = \varnothing$.

Note, that for diffeomorphisms of the first class ($\lambda_1 > 0$, $\lambda_2 > 0$, $d_{21} < 0$, $c_{21} < 0$) inequality (3.3) is fulfilled for all $i \geqslant \bar{k}_1$, $j \geqslant \bar{k}_2$; i.e., in this case $T_{21}(\sigma_j^{12}) \cap \sigma_i^{01} = \varnothing$ for all i and j large enough. (In fact, the horseshoes $T_{21}(\sigma_j^{12})$ and the strips σ_i^{01} lie in different components of $\Pi_1^+ \setminus W^s_{\mathrm{loc}}(O_1)$.) On the other hand, for diffeomorphisms of the second class ($\lambda_1 > 0$, $\lambda_2 > 0$, $d_{21} > 0$, $c_{21} < 0$) inequality (3.2) is fulfilled for all $i \geqslant \bar{k}_1$, $j \geqslant \bar{k}_2$; i.e., in this case the intersections of the horseshoes $T_{21}(\sigma_j^{12})$ and the strips σ_i^{01} are regular for all sufficiently large i and j.

Unlike two above cases, diffeomorphisms of the third class can possess also irregular intersections of the horseshoes and the strips. This implies a possibility of the existence of nonhyperbolic periodic orbits. The main attention will be give to single-round periodic orbits of such a type.

Recall, that a single-round periodic orbit has exactly one intersection point with every of neighbourhoods Π_s^+ and Π_s^-, $s = 1, 2$. Let

$$\begin{aligned} &M_{01}(x_{01}, y_{01}) \in \sigma_i^{01}, \qquad M_{11}(x_{11}, y_{11}) \in \sigma_i^{11} \\ &M_{02}(x_{02}, y_{02}) \in \sigma_j^{02}, \qquad M_{12}(x_{12}, y_{12}) \in \sigma_j^{12} \end{aligned} \tag{3.4}$$

be intersection points of the orbit with the corresponding strips where $i \geqslant \bar{k}_1$, $j \geqslant \bar{k}_2$. Then, $M_{01} = T_{01}^i(M_{01})$, $M_{02} = T_{12}(M_{11})$, $M_{12} = T_{02}^j(M_{02})$, $M_{01} = T_{21}(M_{12})$. Thus, the point M_{01} is a fixed point of the following map

$$T_{ij} \equiv T_{21}\,T_{02}^j\,T_{12}\,T_{01}^i : \sigma_i^{01} \to \Pi_1^+$$

In virtue of (3.1), coordinates of the points M_{01} and M_{11} satisfy equations

$$\begin{aligned} x_{11} &= \lambda_1^i x_{01}(1 + i u_1^{(1)} \lambda_1^i x_{01} y_{11}) + O(\lambda_1^{2i}) \\ y_{01} &= \lambda_1^i y_{11}(1 + i u_1^{(1)} \lambda_1^i x_{01} y_{11}) + O(\lambda_1^{2i}) \end{aligned} \tag{3.5}$$

and coordinates of the points M_{02} and M_{12} satisfy equations

$$\begin{aligned} x_{12} &= \lambda_2^j x_{02}(1 + ju_1^{(2)}\lambda_2^j x_{02} y_{12}) + O(\lambda_2^{2j}) \\ y_{02} &= \lambda_2^j y_{12}(1 + ju_1^{(2)}\lambda_2^j x_{02} y_{12}) + O(\lambda_2^{2j}) \end{aligned} \tag{3.6}$$

Thus, in virtue of (1.2) and (1.3), the map T_{ij} can be written in the form

$$\begin{aligned} &\bar{x}_{02} - x_2^+ \\ &\quad = a_{12}\lambda_1^i x_{01}(1 + iu_1^{(1)}\lambda_1^i x_{01} y_{11}) + b_{12}(y_{11} - y_1^-) + h_1(x_{01}, y_{11}) \\ &\lambda_2^j \bar{y}_{12}(1 + ju_1^{(2)}\lambda_2^j \bar{x}_{02}\bar{y}_{12} + O(\lambda_2^j)) \\ &\quad = c_{12}\lambda_1^i x_{01}(1 + iu_1^{(1)}\lambda_1^i x_{01} y_{11}) + d_{12}(y_{11} - y_1^-) + h_2(x_{01}, y_{11}) \\ &\bar{x}_{01} - x_1^+ \\ &\quad = a_{21}\lambda_2^j x_{02}(1 + ju_1^{(2)}\lambda_2^j \bar{x}_{02}\bar{y}_{12}) + b_{21}(\bar{y}_{12} - y_2^-) + h_3(\bar{x}_{02}, \bar{y}_{12}) \\ &\lambda_1^i \bar{y}_{11}(1 + iu_1^{(1)}\lambda_1^i \bar{x}_{01}\bar{y}_{11} + O(\lambda_1^i)) \\ &\quad = c_{21}\lambda_2^j \bar{x}_{02}(1 + ju_1^{(2)}\lambda_2^j \bar{x}_{02}\bar{y}_{12}) + d_{21}(\bar{y}_{12} - y_2^-) + h_4(\bar{x}_{02}, \bar{y}_{12}) \end{aligned} \tag{3.7}$$

where

$$h_{1,2} = O[(|\lambda_1^i x_{01}| + |y_{11} - y_1^-|)^2], \qquad h_3 = O[(|\lambda_2^j \bar{x}_{02}| + |\bar{y}_{12} - y_2^-|)^2]$$
$$h_4 = O[\lambda_2^{2j}\bar{x}_{02}^2 + |\lambda_2^j \bar{x}_{02}|\,|\bar{y}_{12} - y_2^-| + |\bar{y}_{12} - y_2^-|^3]$$

Lemma 4. The map T_{ij}, by means of a linear change of coordinates, can be brought to a C^{r-1}-smooth area-preserving map of the following form

$$\begin{aligned} \bar{X} &= Y + \varphi_{ij}^1(X, Y) \\ \bar{Y} &= M_{ij} - Y^2 + A_{ij}Y - X + \varphi_{ij}^2(X, Y) \end{aligned} \tag{3.8}$$

where

$$\begin{aligned} A_{ij} &= b_{21}c_{12}\lambda_2^{-j}\lambda_1^i + b_{12}c_{21}\lambda_1^{-i}\lambda_2^j \\ M_{ij} &= d_{12}d_{21}^2\lambda_1^{-2i}\lambda_2^{-2j}(1+\alpha_{ij})[c_{21}\lambda_2^j x_2^+(1 + j\lambda_2^j u_1^{(2)} x_2^+ y_2^-) \\ &\quad - \lambda_1^i y_1^-(1 + i\lambda_1^i u_1^{(1)} x_1^+ y_1^-) + O(\lambda_1^{2i} + \lambda_2^{2j})] \end{aligned} \tag{3.9}$$

and $\varphi_{ij} \to 0$, $\alpha_{ij} \to 0$ as $i, j \to \infty$. In addition, the domain of definition of map (3.8) contains the rectangle $|X| \leqslant C_0\,|\lambda_1|^{-i}\,|\lambda_2|^{-j}$, $|Y| \leqslant C_0\,|\lambda_1|^{-i}\,|\lambda_2|^{-2j}$ where C_0 is a positive constant independent of i and j.

Proof. Introduce the following coordinates

$$x_{01} - x_1^+ = \xi_1, \qquad x_{02} - x_2^+ = \xi_2, \qquad y_{11} - y_1^- = \eta_1, \qquad y_{12} - y_2^- = \eta_2 \tag{3.10}$$

where

$$|\xi_1| \leqslant \varepsilon_1^+, \qquad |\eta_1| \leqslant \varepsilon_1^-, \qquad |\xi_2| \leqslant \varepsilon_2^+, \qquad |\eta_2| \leqslant \varepsilon_2^- \tag{3.11}$$

and ε_s^+ and ε_s^- are diameters of the neighbourhoods $\Pi_s^+ \Pi_s^-$, respectively, $s = 1, 2$.

In virtue of (3.7), the map T_{ij} has the following form in coordinates (3.10)

$$\begin{aligned}
\bar{\xi}_2 &= a_{12}\lambda_1^i \xi_1 + b_{12}\eta_1 \\
&\quad + a_{12}\lambda_1^i x_1^+(1 + i\lambda_1^i x_1^+ y_1^-(1 + \cdots) \\
&\quad + O(i\lambda_1^{2i}(|\xi_1| + |\eta_1|) + \eta_1^2) \\
\lambda_2^j \bar{\eta}_2(1 + j\lambda_2^j O(|\bar{\xi}_2| + |\bar{\eta}_2|)) &= c_{12}\lambda_1^i \xi_1 + d_{12}\eta_1 + c_{12}\lambda_1^i x_1^+(1 + \cdots) \\
&\quad - \lambda_2^j y_2^-(1 + j\lambda_2^j x_2^+ y_2^-(1 + \cdots)) \\
&\quad + O(i\lambda_1^{2i}(|\xi_1| + |\eta_1|) + \eta_1^2) \\
\bar{\xi}_1 &= a_{21}\lambda_2^j \bar{\xi}_2 + b_{21}\bar{\eta}_2 \\
&\quad + a_{21}\lambda_2^j x_2^+(1 + j\lambda_2^j x_2^+ y_2^-(1 + \cdots)) \\
&\quad + O(j\lambda_2^{2j}(|\bar{\xi}_2| + |\bar{\eta}_2|) + \bar{\eta}_2^2) \\
\lambda_1^i \bar{\eta}_1(1 + i\lambda_1^i O(|\bar{\xi}_1| + |\bar{\eta}_1|)) &= c_{21}\lambda_2^j \bar{\xi}_2 + d_{21}\bar{\eta}_2^2 + c_{21}\lambda_2^j x_2^+(1 + j\lambda_2^j x_2^+ y_2^- u_1^{(2)}) \\
&\quad - \lambda_1^i y_1^-(1 + i\lambda_1^i x_1^+ y_1^- u_1^{(1)}) \\
&\quad + O(\lambda_1^{2i} + \lambda_2^{2j}) + O(j\lambda_2^{2j}(|\bar{\xi}_2| + |\bar{\eta}_2|) + \bar{\eta}_2^3)
\end{aligned} \tag{3.12}$$

where we denote by ellipsis, here and below at the proving the lemma, terms which are independent of coordinates and tend to zero as $i, j \to \infty$.

Shifting coordinates

$$\xi_2 \to \xi_2 - a_{12}\lambda_1^i x^+{}_1 + \frac{b_{12}}{d_{12}}(c_{12}\lambda_1^i x_1^+ - \lambda_2^j y_2^-)(1 + \cdots)$$

$$\xi_1 \to \xi_1 - a_{21}\lambda_2^j x_2^+(1 + \cdots)$$

$$\eta_1 \to \eta_1 + \frac{1}{d_{12}}(c_{12}\lambda_1^i x_1^+ - \lambda_2^j y_2^-)(1 + \cdots)$$

brings (3.12) to the form

$$\begin{aligned}
\bar{\xi}_2 &= a_{12}\lambda_1^i \xi_1 + b_{12}\eta_1 + O(i\lambda_1^{2i}(|\xi_1| + |\eta_1|) + |\eta_1|)^2) \\
\lambda_2^j \bar{\eta}_2(1 + j\lambda_2^j O(|\bar{\xi}_2| + |\bar{\eta}_2|)) &= c_{12}\lambda_1^i \xi_1 + d_{12}\eta_1 + O(i\lambda_1^{2i}(|\xi_1| + |\eta_1|) + |\eta_1|)^2) \\
\bar{\xi}_1 &= a_{21}\lambda_2^j \bar{\xi}_2 + b_{21}\bar{\eta}_2 + O(j\lambda_2^j(|\bar{\xi}_2| + |\bar{\eta}_2|) + |\bar{\eta}_2^2) \\
\lambda_1^i \bar{\eta}_1(1 + i\lambda_1^i O(|\bar{\xi}_1| + |\bar{\eta}_1|)) &= c_{21}\lambda_2^j \bar{\xi}_2 + d_{21}\bar{\eta}_2^2 + \mu_1 \\
&\quad + O(j\,|\lambda_2^j\,|(|\bar{\xi}_2| + |\bar{\eta}_2|) + |\bar{\eta}_2|^3)
\end{aligned} \tag{3.13}$$

where

$$\mu_1 = [c_{21}\lambda_2^j x_2^+(1 + j\lambda_2^j u_1^{(2)} x_2^+ y_2^-) - \lambda_1^i y_1^-(1 + i\lambda_1^i u_1^{(1)} x_1^+ y_1^-) + O(\lambda_1^{2i} + \lambda_2^j)]$$

We solve now from the first and second equations of (3.13) coordinates $\bar{\xi}_2$ and $\bar{\eta}_2$ with respect to ξ_1 and η_1:

$$\begin{aligned}
\bar{\xi}_2 &= a_{12}\lambda_1^i \xi_1 + b_{12}\eta_1 + O(i\lambda_1^{2i}(|\xi_1| + |\eta_1|) + |\eta_1|)^2) \\
\bar{\eta}_2 &= \lambda_2^{-j}[c_{12}\lambda_1^i \xi_1 + d_{12}\eta_1 + O\{j\,|\lambda_2|^j\,(|\eta_1| + |\lambda_1^i \xi_1| + \eta_1^2)\}]
\end{aligned} \tag{3.14}$$

Consider the following linear change of coordinates

$$u = \frac{1}{d_{12}}\,\xi_1, \qquad v = c_{12}\lambda_1^i \xi_1 + d_{12}\eta_1 \tag{3.15}$$

We have from here

$$\xi_1 = d_{12}u, \qquad \eta_1 = \frac{1}{d_{12}}\,v - c_{12}\lambda_1^i u$$

In virtue of (3.14) and (3.15), coordinates $\bar{\xi}_2$ and $\bar{\eta}_2$ depends on u and v in the following way

$$\begin{aligned}
\bar{\xi}_2 &= a_{12}\lambda_1^i \xi_1 + b_{12}\eta_1 + \cdots = a_{12}d_{12}\lambda_1^i u + \frac{b_{12}}{d_{12}}\,v - b_{12}c_{12}\lambda_1^i u) + \cdots \\
&= \lambda_1^i u(a_{12}d_{12} - b_{12}c_{12}) + \frac{b_{12}}{d_{12}}\,v + \cdots = \lambda_1^i u\,\frac{b_{12}}{d_{12}}\,v + O(i\lambda_1^{2i}(|u| + |v|) + v^2) \\
\bar{\eta}_2 &= \lambda_2^{-j}[v + O\{j\,|\lambda_2|^j\,(|v| + |\lambda_1|^i\,|u|) + v^2 + |\lambda_1|^i\,|u|\,|v|\}]
\end{aligned}$$

Using these relations we bring T_{ij} to the following form (in coordinates u and v):

$$\begin{aligned}\bar{u} &= \frac{b_{21}}{d_{12}}\lambda_2^{-j}[v + O\{j\,|\lambda_2|^j\,(|v| + |\lambda_1|^i\,|u|) + v^2 + |\lambda_1|^i\,|u|\,|v|\}\,] \\ \bar{v} &= c_{12}\lambda_1^i\bar{\xi}_1 + d_{12}\bar{\eta}_1 \\ &= d_{12}d_{21}\lambda_1^{-i}\lambda_2^{-2j}[v + O\{j\,|\lambda_2|^j\,(|v| + |\lambda_1|^i\,|u|) + v^2 + |\lambda_1|^i\,|u|\,|v|\}\,]^2 \\ &\quad + v(b_{21}c_{12}\lambda_2^{-j}\lambda_1^i + b_{12}c_{21}\lambda_1^{-i}\lambda_2^j) + d_{12}c_{21}\lambda_2^j u(1 + O(i\lambda_1^i u)) + \mu_2\end{aligned} \tag{3.16}$$

where $\mu_2 = d_{12}\lambda_1^{-i}(1 + \cdots)\,\mu_1$.

Now, we rescale the coordinates in the following way

$$u = -\frac{b_{21}}{d_{12}^2 d_{21}}\lambda_1^i\lambda_2^j X, \qquad v = -\frac{1}{d_{12}d_{21}}\lambda_1^i\lambda_2^{2j}Y \tag{3.17}$$

Then, the map (3.16) is written in the form

$$\begin{aligned}\bar{X} &= Y + \cdots \\ \bar{Y} &= -Y^2 + Y(b_{21}c_{12}\lambda_2^{-j}\lambda_1^i + b_{12}c_{21}\lambda_1^{-i}\lambda_2^j) + b_{21}c_{21}X + M_{ij} + \cdots \\ &= -Y^2 + A_{ij}Y - X + M_{ij} + \cdots\end{aligned} \tag{3.18}$$

where formula (3.9) takes place for A_{ij} and M_{ij} and the dots stand for terms tending to zero as $i, j \to \infty$.

We note finally, that the map T_{ij}, written in coordinates (X, Y), has a big domain of definition. Really, it follows from (3.11), (3.15) and (3.17) that variables X and Y may take any values from the following intervals $|X| \leqslant C_1\,|\lambda_1|^{-i}\,|\lambda_2|^{-j}\,\varepsilon_1^+$ and $|Y| \leqslant C_1\,|\lambda_1|^{-i}\,|\lambda_2|^{-2j}\,\varepsilon_1^-$ where C_1 is a positive constant. This completes the proof of the lemma.

It is seen from (3.9) that the rescaling parameter M_{ij} can take any finite values only for those i and j at which the quantity $c_{21}\lambda_2^j x_2^+ - \lambda_1^i y_1^-$ can be made how much small in order to compensate the how much big factor $\lambda_1^{-2i}\lambda_2^{-2j}$. By Lemma 3 it is possible only for those i and j at which the intersection of the horseshoe $T_{21}\sigma_j^{12}$ with the strip σ_i^{01} is irregular. Below we will consider only those i and j which satisfy the inequality

$$|c_{21}\lambda_2^j x_2^+ - \lambda_1^i y_1^-| \leqslant |\lambda_1|^{i/2}\,|\lambda_1|^i \tag{3.19}$$

First of all, we will prove that inequality (3.19) can have infinitely many integer solutions with respect to i and j. For this end, we rewrite (3.19) in the following form

$$\lambda_1^i y_1^-(1-\lambda_1^{i/2}/y_1^-) \leqslant c_{21}\lambda_2^j x_2^+ \leqslant \lambda_1^i y_1^-(1+\lambda_1^{i/2}/y_1^-)$$

and take the logarithm. As result, we obtain the inequality

$$|i - j\theta + \tau| \leqslant \frac{|\lambda_1|^{i/2}}{y_1^- \ln |\lambda_1^{-1}|} \tag{3.20}$$

which is equivalent to (3.19) (modulo terms in (3.20) of the order $O(|\lambda_1|^i)$). Here, θ and τ are the quantities from (1.5) and (1.6).

Lemma 5. For any τ and any functions v_{ij}^1 and v_{ij}^2 which are continuous in θ and τ and such that $v_{ij}^1 < v_{ij}^2$ and v_{ij}^1, $v_{ij}^2 \to 0$ as $i, j \to \infty$, the inequality

$$v_{ij}^1 < i - j\theta + \tau < v_{ij}^2 \tag{3.21}$$

has infinitely many integer-valued solutions for a set of values of θ which is dense in R^1.

Apparently, this result is well known in the arithmetical number theory (in any case, similar results are in refs. 33 and 34), but we give its proof for completeness. Let us show that in the interval $\delta_0 = (\theta_0 - \varepsilon, \theta_0 + \varepsilon)$, for any θ_0 and $\varepsilon > 0$, there exist such values of θ that inequality (3.21) has infinitely many integer-valued solutions. Let $\theta_1 \in \delta_0$ and $\theta_1 > \theta_0$. Then, the straight lines $y - x\theta_1 + \tau = 0$ and $y - i\theta_0 + \tau = 0$ form an angle inside which a countable set of points with integer-valued coordinates lies. Since $v_{ij} \to 0$ as $i, j \to \infty$ and v_{ij} are continuous in θ, there exists an interval $\delta_1 \subset (\theta_0, \theta_1) \subset \delta_0$ such that inequality (3.21) has an integer-valued solution $(i = i_1, j = j_1)$ for every $\theta \in \delta_1$. Analogously, in the interval δ_1 we find a subinterval δ_2 such that for $\theta \in \delta_2$ inequality (3.21) has now two integer-valued solutions (i_1, j_1) and (i_2, j_2). Acting in this way, we obtain an infinite sequence of embedded intervals $\delta_0 \supset \delta_1 \supset \cdots \supset \delta_n \supset \cdots$ such that inequality (3.21) has n integer-valued solutions $(i_1, j_1), (i_2, j_2), \ldots, (i_n, j_n)$ for $\theta \in \delta_n$. Let θ^* be a value of θ such that $\theta^* \in \delta_n$ for any n, then inequality (3.21) has infinitely many integer-valued solutions at $\theta = \theta^*$. This completes the proof of the lemma in virtue of the arbitrariness of choice of initial θ_0 and ε.

By Lemma 5, inequality (3.20) (and, hence, (3.19)) has solutions with how much large i and j.

Lemma 6. Let i and j satisfy (3.19). Then, M_{ij} takes values from the interval

$$(-C_2\lambda_2^{-2j}\,|\lambda_1|^{-i/2},\, C_2\lambda_2^{-2j}\,|\lambda_1|^{-i/2}) \tag{3.22}$$

where C_2 is a positive constant independent of i and j, and A_{ij} is represented in the form

$$A_{ij} = \frac{b_{12}y_1^-}{x_2^+} - \frac{c_{12}x_2^+}{y_1^-} + O(|\lambda_1|^{i/2}) \tag{3.23}$$

Proof. If to substitute into (3.9) the boundaries of interval (3.19), then we obtain inequality (3.22).

It follows from (3.19) that

$$|y_1^- - c_{21}x_2^+\lambda_2^j\lambda_1^{-i}| \leqslant |\lambda_1|^{i/2}$$

and, hence, for such i and j the following equality takes place

$$\lambda_2^j\lambda_1^{-i} = \frac{y_1^-}{c_{21}x_2^+} + O(|\lambda_1|^{i/2}) \tag{3.24}$$

Thus,

$$\begin{aligned} A_{ij} &= b_{21}c_{12}\lambda_1^i\lambda_2^{-j} + b_{12}c_{21}\lambda_2^j\lambda_1^{-i} \\ &= b_{21}c_{12}\frac{c_{21}x_2^+}{y_1^-} + b_{12}c_{21}\frac{y_1^-}{c_{21}x_2^+} + O(|\lambda_1|^{i/2}) \\ &= \frac{b_{12}y_1^-}{x_2^+} - \frac{c_{12}x_2^+}{y_1^-} + O(|\lambda_1|^{i/2}) \end{aligned}$$

The lemma is proved.

Lemma 7. The quantities

$$\frac{b_{12}y_1^-}{x_2^+}, \qquad \frac{c_{12}x_2^+}{y_1^-}, \qquad \frac{c_{21}x_1^+}{y_2^-} \tag{3.25}$$

do not depend on choice of pairs of heteroclinic points of the orbits Γ_{12} and Γ_{21}.

Proof. Note, that the first two quantities in (3.25) depend only on coefficients of the global map T_{12}. Hence, it is sufficiently to check the invariance of these quantities with respect to choice of pairs of heteroclinic points of the orbit Γ_{12}. It is clear also that for this end it is sufficiently to prove the invariance with respect to choice of the following pairs of heteroclinic points: (a) $T_{01}^{-1}(M_1^-)$ and M_2^+; and (b) M_1^- and $T_{02}M_2^+$.

In the case (a), the map $T'_{12} \equiv T_{12}T_{01} : T_{01}^{-1}\Pi_1^- \to \Pi_2^+$ will play a role of the global map T_{12}. In virtue of (2.6) and (1.2), T'_{12} can be written as follows

$$\bar{x}_{02} - x_2^+ = a_{12}\lambda_1 x'_{11} + \lambda_1^{-1} b_{12}(y'_{11} - \lambda_1 y_1^-) + \cdots$$

$$\bar{y}_{02} = c_{12}\lambda_1 x'_{11} + \lambda_1^{-1} d_{12}(y'_{11} - \lambda_1 y_1^-) + \cdots$$

Thus, $b'_{12} = \lambda_1^{-1} b_{12}$, $c'_{12} = \lambda_1 c_{12}$, $y_1^{-\prime} = \lambda_1 y_1^-$, $x_2^{+\prime} = x_2^+$. This implies that

$$\frac{b'_{12} y_1^{-\prime}}{x_2^{+\prime}} = \frac{\lambda_1^{-1} b_{12}\lambda_1 y_1^-}{x_2^+} = \frac{b_{12} y_1^-}{x_2^+}$$

$$\frac{c'_{12} x_2^{+\prime}}{y_1^{-\prime}} = \frac{\lambda_1 c_{12} x_2^+}{\lambda_1 y_1^-} = \frac{c_{12} x_2^+}{y_1^-}$$

In the same way there is proved the invariance of the quantities $b_{12} y_1^-/x_2^+$ and $c_{12}x_2^+/y_1^-$ in the case (b). Here, a role of T_{12} will be played by the map $T'_{12} \equiv T_{02}T_{12} : \Pi_1^- \to T_{02}\Pi_1^+$, for which $b'_{12} = \lambda_2 b_{12}$, $c'_{12} = \lambda_2^{-1} c_{12}$, $y_1^{-\prime} = y_1^-$, $x_2^{+\prime} = \lambda_2 x_2^+$.

The quantity $c_{21}x_1^+/y_2^-$ (the third quantity in (3.25)) depends only on coefficients of the map T_{21}. Its value is not changed if to choose instead the pair M_2^- and M_1^+ the following ones: (a) $T_{02}^{-1}(M_2^-)$ and M_1^+; and (b) M_2^- and $T_{01}M_1^+$. Indeed, in the case (a) $T'_{21} \equiv T_{21}T_{02} : T_{02}^{-1}\Pi_2^- \to \Pi_1^+$ and, in virtue of Lemma 1 and (1.3),

$$c'_{21} = \lambda_2 c_{21}, \qquad y_2^{-\prime} = \lambda_2 y_2^-, \qquad x_1^{+\prime} = x_1^+$$

In the case (b) $T'_{21} \equiv T_{01}T_{21} : \Pi_2^- \to T_{01}\Pi_1^+$ and

$$c'_{21} = \lambda_1^{-1} c_{21}, \qquad y_2^{-\prime} = y_2^-, \qquad x_1^{+\prime} = \lambda_1 x_1^+$$

Evidently, that the value of quantity $c'_{21}x_1^{+\prime}/y_2^{-\prime}$ is equal to $c_{21}x_1^+/y_2^-$ in both cases. The lemma is proved.

These lemmas show that, for a countable set of integers i and j (satisfying (3.20)), the map T_{ij} is close to the following map $\mathscr{H}_M$

$$\begin{aligned}\bar{x} &= y \\ \bar{y} &= M - y^2 + Ay - x\end{aligned} \tag{3.26}$$

where $A = (b_{12} y_1^- / x_2^+) - (c_{12} x_2^+ / y_1^-)$ (Lemma 6) and coordinates (x, y) and parameter M can take arbitrary finite values (Lemmas 4 and 6). Note, that map $\mathscr{H}_M$, by shifting origin and changing the parameter, can be brought to the following standard conservative Henon map

$$\bar{x} = y, \qquad \bar{y} = 1 - ay^2 - x$$

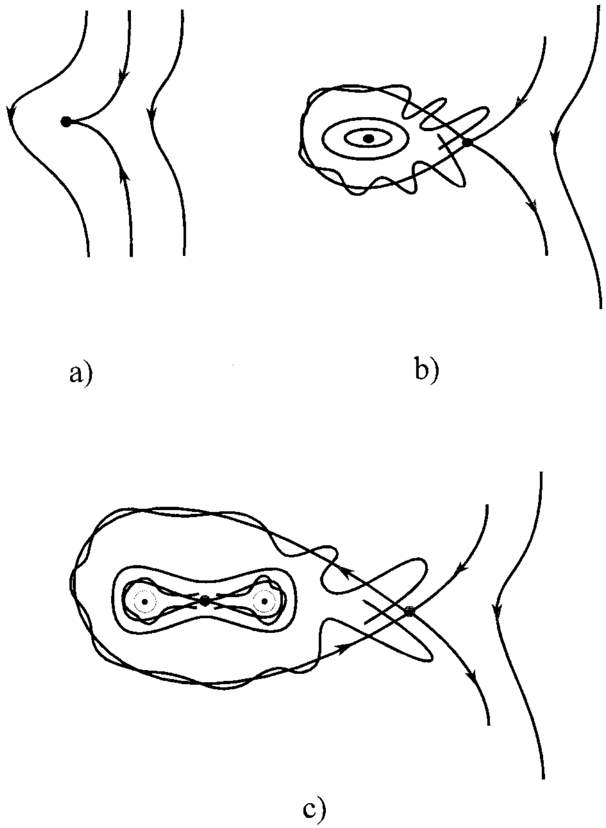

Fig. 8. The structure of fixed points for the conservative Henon map: (a) the parabolic point ($\nu_1 = \nu_2 = +1$) at $M = M_0$; (b) the saddle and elliptic fixed points at $M_0 < M < M_\pi$; (c) two saddle fixed points and elliptic cycle of period 2 at $M > M_\pi$.

The map $\mathscr{H}_M$ is remarkable in that sense that bifurcations of its fixed points are well known. Namely, $\mathscr{H}_M$ has no fixed points at $M < M_0$, where $M_0 = -(A-2)^2/4$; it has a parabolic fixed point (unstable) with eigenvalues $\nu_1 = \nu_2 = 1$ at $M = M_0$ (Fig. 8a) and a parabolic fixed point (stable) with eigenvalues $\nu_1 = \nu_2 = -1$ at $M = M_\pi = (A+2)(6-A)/4$; at $M > M_\pi$ the last point becomes the saddle-minus fixed point (with negative eigenvalues) in whose neighbourhood an elliptic point of period 2 is appeared (Fig. 8c). Especially, we turn our attention to the fact that $\mathscr{H}_M$ has an elliptic fixed point (Fig. 8b) at $M_0 < M < M_\pi$ (its eigenvalues are $\nu_{1,2} = e^{\pm i\psi}$ at $M = M_\psi = \frac{1}{4}(A - 2\cos\psi)(4 - A - 2\cos\psi)$). It was established in ref. 35 that in a neighbourhood of this point (if $\psi \notin \{\pi/2, 2\pi/3\}$) the map $\mathscr{H}_M$ can be written in the complex form (1.4) where

$$h(\psi) = -2\,\frac{(1+\cos\psi)(4\cos\psi + 1)}{(2\cos\psi + 1)\sin\psi}$$

Hence, the coefficient $h(\psi)$ is not equal to zero if $\psi \neq \arccos(-1/4)$. Therefore, according to the KAM-theory, the pointed out fixed point is generic, except for the cases where $\psi \in \{\pi/2, \arccos(-1/4), 2\pi/3\}$.

4. PROOFS OF THE THEOREMS

As we noted in Section 1, there exist seven types of diffeomorphisms of the third classes (see Table 1). We will prove Theorems 1, 2 and 3 only for the case of diffeomorphisms from H_3^1 (i.e., for the case where parameters λ_1, λ_2, c_{21} and d_{21} are positive), since in other cases the proofs are rather analogously.

Let i and j satisfy inequality (3.19). As it follows from Lemmas 4–7, in this case the map T_{ij} is brought to a C^{r-1}-smooth area-preserving map of the following form

$$\begin{aligned} \bar{x} &= y + \tilde{\varphi}^1_{ij}(x, y) \\ \bar{y} &= M_{ij} - y^2 + Ay - x + \tilde{\varphi}^2_{ij}(x, y) \end{aligned} \tag{4.1}$$

where functions $\tilde{\varphi}_{ij}(x, y)$ tend to zero together with derivatives (up to order $(r-2)$) as $i, j \to \infty$; $A = b_{12} y_1^- / x_2^+ - c_{12} x_2^+ / y_1^-$ is a constant independent of choice of heteroclinic points (Lemma 7). By Lemma 4 the domain of definition of this map contains the rectangle

$$|x| \leqslant C_0\, |\lambda_1|^{-i}\, |\lambda_2|^{-j}, \qquad |y| \leqslant C_0\, |\lambda_1|^{-i}\, |\lambda_2|^{-2j}$$

Recall also (Lemmas 4 and 6) that the range of values of the parameter M_{ij} contains the interval

$$(-C_2\lambda_2^{-2j}\,|\lambda_1|^{-i/2},\ C_2\lambda_2^{-2j}\,|\lambda_1|^{-i/2})$$

Here C_0 and C_2 are some positive constants independent of i and j.

Note, that the map $\mathscr{H}_M$ has a generic elliptic fixed point at every value of M from the interval $M_0 < M < M_{\pi/2}$ where $M_0 = -\frac{1}{4}(A-2)^2$, $M_{\pi/2} = \frac{1}{4}A(4-A)$. Then, there exists such $\varepsilon_{ij} > 0$, $\varepsilon_{ij} \to 0$ as $i, j \to \infty$, that the map T_{ij} (which can be brought to form (4.1)) has a generic elliptic fixed point for all values of M_{ij} satisfying the inequality $M_0 + \varepsilon_{ij} < M_{ij} < M_{\pi/2} - \varepsilon_{ij}$. In virtue of (3.9), this inequality can be written in the form

$$\frac{M_0}{d_{12}d_{21}^2}\,\lambda_1^{2i}\lambda_2^{2j} < c_{21}\lambda_2^j x_2^+ (1 + j\lambda_2^j u_1^{(2)} x_2^+ y_2^- + \cdots)$$
$$- \lambda_1^i y_1^- (1 + i\lambda_1^i u_1^{(1)} x_1^+ y_1^- + \cdots) < \frac{M_{\pi/2}}{d_{12}d_{21}^2}\,\lambda_1^{2i}\lambda_2^{2j} \quad (4.2)$$

Taking the logarithm of (4.2) we obtain the following inequality

$$\tilde{v}_{ij}^1 < i - j\theta + \tau - \frac{1}{\ln|\lambda_1|}\ln\frac{1 + j\lambda_2^j u_1^{(2)} x_2^+ y_2^-}{1 + i\lambda_1^i u_1^{(1)} x_1^+ y_1^-} < \tilde{v}_{ij}^2 \tag{4.3}$$

where the functions $\tilde{v}_{ij}^1$ and $\tilde{v}_{ij}^2$ depend continuously on θ and τ and are such that

(1) $\tilde{v}_{ij}^\alpha = O(\lambda_1^{2i} + \lambda_2^{2j})$,

(2) $\tilde{v}_{ij}^1 < \tilde{v}_{ij}^2$,

(3) $\tilde{v}_{ij}^2 - \tilde{v}_{ij}^1 \sim (\lambda_1^i \lambda_2^{2j})$.

Note, that for such i and j inequalities (3.20) and (3.19) are fulfilled automatically. Since $i \sim j\theta$ and $c_{21}x_2^+\lambda_2^j \sim \lambda_1^i y_1^-$ in this case, it follows that

$$a_{ij} \equiv \frac{1}{\ln\lambda_1}\ln\frac{1 + j\lambda_2^j u_1^{(2)} x_2^+ y_2^-}{1 + i\lambda_1^i u_1^{(1)} x_1^+ y_1^-} = j\lambda_2^j\,\frac{x_2^+ y_2^-}{\ln\lambda_1}\left(u_1^{(2)} - \theta u_1^{(1)}\,\frac{c_{21}x_1^+}{y_2^-}\right) + O(\lambda_2^j)$$

and inequality (4.2) coincides with (1.8). This completes the proof of Theorem 1.

By Lemma 5, inequality (4.3) has infinitely many integer-valued solutions at a dense (in the plane (θ, τ)) set I of values of θ and τ. Let $(\theta^*, \tau^*) \in I$ and (i_n, j_n), $n = 1, 2, \ldots$, be the corresponding countable set of integer-valued pairs such that inequality (4.3) is fulfilled at $i = i_n$, $j = j_n$. Then, diffeomorphism $T_{\theta^*\tau^*} \in H_3$ possesses infinitely many single-round elliptic periodic orbits each of which has exactly one intersection point with every strip $\sigma^{01}_{i_n} \subset \Pi_1^+$ and $\sigma^{02}_{j_n} \subset \Pi_2^+$. These strips are with different numbers and do not intersect. Hence, the single-round elliptic periodic orbits are isolated and since they are generic, every of these elliptic periodic orbits is surrounded by the own elliptic (periodic) island. Thus, diffeomorphism $T_{\theta^*\tau^*} \in H_3$ possesses infinitely many elliptic periodic islands. Since $i_n, j_n \to \infty$ as $n \to \infty$, the closure of the pointed out countable set of elliptic orbits (islands) contains the saddles O_1 and O_2 and also the heteroclinic orbits Γ_{12} and Γ_{21}. Note, that functions $\tilde{v}^1_{ij}$ and $\tilde{v}^2_{ij}$ (from (4.3)) are, by conditions (1)–(3), exponentially small. This means that if diffeomorphism T has infinitely single-round elliptic periodic orbits, than its invariants θ and τ are connected by strong arithmetical relations: θ and τ must admit "exponentially good nonhomogeneous approximations by rational numbers." This completes the proof of Theorem 2.

By Lemma 7, coefficient $Q = (u_1^{(2)} - \theta u_1^{(1)}(c_{21} x_1^+/y_2^-)$, does not depend on choice of pairs of heteroclinic points. (Moreover, the quantity $x_2^+ y_2^- Q$ does not depend also on rescaling the variables). If $Q \neq 0$, then inequality (4.3) can possess infinitely many integer solutions at irrational θ only. Indeed, for a rational θ the straight line $j = i\theta - \tau$ either lies on a finite distance from the points of the integer-valued lattice or contains a countable set of such points. In the first case, inequality (4.3) can not have solutions for large i and j since $|a_{ij}| + |\tilde{v}^\alpha_{ij}| \to 0$ as $i, j \to \infty$. In the second case, this fact is true also because $v^\alpha_{ij} = o(a_{ij})$, i.e., here the set (on the plane) defined by inequality (4.3) does not contain any points of the straight line $y = x\theta - \tau$.

We will prove now Theorem 3. Let $T \in H_3$. We include T into a one parameter family T_θ of diffeomorphisms on H_3 and assume that $T = T_{\theta_0}$. By Lemma 4, in the case where diffeomorphism T_θ has a parabolic single-round periodic orbit with multipliers $\nu_1 = \nu_2 = +1$ (whose point in Π_1^+ is a fixed parabolic point for the map T_{ij}) the following equality is fulfilled

$$d_{12} d_{21}^2 \lambda_1^{-2i} \lambda_2^{-2j} (1 + \alpha_{ij}) [c_{21} \lambda_2^j x_2^+ - \lambda_1^i y_1^- + O(i\lambda_1^{2i} + j\lambda_2^{2j}] = M_0 + \beta_{ij} \tag{4.4}$$

where $\beta_{ij} \to 0$ as $i, j \to \infty$. Equality (4.4) can be rewritten in the form

$$c_{21} \lambda_2^j x_2^+ (1 + O(j\lambda_2^j) = \lambda_1^i y_1^- (1 + O(i\lambda_1^i) \tag{4.5}$$

If to take the logarithm of (4.5), then one can easily found the corresponding value of $\theta = \theta_{ij}^{+1}$. Namely,

$$\theta_{ij}^{+1} = \frac{i}{j} + \frac{\tau}{i} + O(i\lambda_1^i + j\lambda_2^j) \tag{4.6}$$

Evidently, such values of θ are dense in an interval $(\theta_0 - \varepsilon, \theta_0 + \varepsilon)$ for any $\varepsilon > 0$.

It is shown analogously that values $\theta = \theta_{ij}^{-1}$ are dense in the interval where θ_{ij}^{-1} is a value of θ at which the map T_{ij} possesses a parabolic fixed point with eigenvalues $\nu_1 = \nu_2 = -1$. In this case relations are fulfilled which differ from (4.6) in terms of the order $O(\lambda_1^i \lambda_2^j)$.

This work is supported in part by RFFI-grant No. 99-01-00231, INTAS-grant No. 97-804 and scientific program "Russian University"—project No. 1905.

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